

SPHERICAL 2-DUPIN SUBMANIFOLDS

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ABSTRACT. We show that every spherical 2-Dupin submanifold that is not a hypersurface is conformally congruent to the standard embedding of the real, complex, quaternionic or octonionic projective plane. We also classify 2-CPC, 2-umbilical and weakly 2-umbilical submanifolds in space forms.

1. INTRODUCTION

An isometric immersion $f : M^n \rightarrow \tilde{M}^m$ is said to be *umbilical* at $x \in M^n$ if there exists η in its normal space $N_f M(x)$ such that its second fundamental form

$$\alpha(X, Y) = \langle X, Y \rangle \eta$$

for all $X, Y \in T_x M$. Clearly, in this case η is the mean curvature vector $\mathcal{H}(x)$ of f at x . Equivalently, f is umbilical at x if its shape operator

$$(1.1) \quad A_\xi = \langle \mathcal{H}(x), \xi \rangle I$$

for every $\xi \in N_f M(x)$. A submanifold is called *umbilical* if it is umbilical at every point.

An isometric immersion $f : M^n \rightarrow \tilde{M}^m$ is said to have *parallel mean curvature vector field* if

$$(1.2) \quad \nabla_X^\perp \mathcal{H} = 0$$

for all $x \in M^n$ and $X \in T_x M$.

In particular, if f has parallel mean curvature vector field then it follows that $\|\mathcal{H}\|$ is constant along M^n .

An immersion is called an *extrinsic sphere* if it is umbilical and the mean curvature vector field is parallel.

It is a well-known fact that every n -dimensional (complete) extrinsic sphere of the space form \mathbb{Q}_c^m is isometric to \mathbb{Q}_c^n , where $c = \tilde{c} + \|\mathcal{H}\|^2$, and it is always a hypersurface in the sense that its codimension can be reduced to one, i.e., it is contained in a totally geodesic submanifold $\mathbb{Q}_{\tilde{c}}^{n+1}$. Moreover, it is uniquely determined by and can be explicitly described in terms of one of its points together with its corresponding tangent space and mean curvature vector. Indeed, in Euclidean space, for instance, apart from affine subspaces (totally geodesic submanifolds), any extrinsic sphere is an actual round sphere, while in the spherical and hyperbolic space forms (regarded as round spheres of the Euclidean and Lorentzian space, respectively) an extrinsic sphere is always the intersection with an affine subspace of the ambient

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Euclidean or Lorentzian space, respectively. Of course an extrinsic sphere is totally geodesic if and only if $\mathcal{H} = 0$. We refer to Dajczer [1] or Berndt-Console-Olmos [2] for the many details of this result.

It is also well known that the parallelism (1.2) of the mean curvature vector field is automatically satisfied for n -dimensional umbilical submanifolds in space forms with $n \geq 2$. This is a simple consequence of the Codazzi equation. In other words, any n -dimensional umbilical submanifold of a space form is always an extrinsic sphere if $n \geq 2$. On the other hand, this is clearly not true if $n = 1$, since every curve is trivially umbilical.

We now discuss how the definitions of umbilical submanifolds and extrinsic spheres can be generalized to encompass some of the most beautiful and widely studied classes of submanifolds, but before proceeding we agree that manifolds are always connected throughout the article and we denote by

$$UN_f^{m-1} = \{(x, \xi) \in N_f M : \|\xi\| = 1\}$$

the unit normal bundle of a given isometric immersion $f : M^n \rightarrow \tilde{M}^m$, except when f is an oriented hypersurface, in which case the right-hand side of the above equation is not connected and so UN_f^n will stand for one of its two connected components. We also use the natural identification of $T_{(x, \xi)} UN_f$ as a subspace of $T_x M \times T_{f(x)} \tilde{M}$ by means of differentiation with respect to the Levi-Civita connection $\tilde{\nabla}$ of \tilde{M} , i.e., given a curve $(x(t), \xi(t))$ in UN_f^{m-1} , we identify

$$(1.3) \quad (x, \xi)' \simeq (x', \tilde{\nabla}_{x'} \xi) = (x', -f_* A_\xi x' + \nabla_{x'}^\perp \xi),$$

where we have used the Weingarten formula in the equality. Since $\xi(t)$ is a unit vector field along $x(t)$, it follows that $\nabla_{x'}^\perp \xi$ is orthogonal to ξ , and hence

$$T_{(x, \xi)} UN_f \simeq \{(X, -f_* A_\xi X + \eta) : X \in T_x M \text{ and } \eta \in N_f M(x), \eta \cdot \xi = 0\}.$$

Note that equation (1.1) can be simply translated into the fact that f has exactly one principal curvature with respect to any $(x, \xi) \in UN_f^{m-1}$. It is hence quite natural to extend this definition to the case in which f has precisely k principal curvatures with respect to every $(x, \xi) \in UN_f^{m-1}$. An isometric immersion $f : M^n \rightarrow \tilde{M}^m$ is said to be *k-umbilical* at $x \in M^n$ if it has exactly k distinct principal curvatures with respect to all $(x, \xi) \in UN_f^{m-1}$. A submanifold is called *k-umbilical* if it is *k-umbilical* at every point. In this case, each principal curvature $\kappa_i : UN_f^{m-1} \rightarrow \mathbb{R}$, $1 \leq i \leq k$, has constant multiplicity and is then smooth as well as its corresponding principal distribution

$$E_i(x, \xi) = \{X \in T_x M : A_\xi X = \kappa_i X\},$$

by the general theory of symmetric smooth tensors on Riemannian manifolds (see, for example, Cecil-Ryan [3], Nomizu [4] or Ryan [5]). Of course, E_i is not a distribution in the strict sense of the word, although we will soon associate it to its horizontal lift T_i , which is then literally a distribution on UN_f^{m-1} . Notice also that, if $k \geq 2$ and $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ is *k-umbilical* at some point x , the definition forces f to be conformally substantial, since in this case no umbilical normal direction at x will then exist.

The definition of an extrinsic sphere, on the other hand, is a little bit more complex, since it involves not only the algebraic nature of the second fundamental form, but also the differential requirement that the mean curvature vector field be

parallel. The attempt to generalize the latter condition to k -umbilical submanifolds, depending on how we look at it, will lead us through a nested sequence

$$(1.4) \quad \mathfrak{U}_k \subset \mathfrak{I}_k \subset \mathfrak{D}_k$$

of three remarkable classes of submanifolds.

Let S^n be an extrinsic sphere in a space form. As discussed at the beginning of the section, we can assume that S^n is an orientable hypersurface. Then the whole information of its second fundamental form is contained in its shape operator A with respect to one of its two unit normal vector fields and condition (1.2) reduces to the constancy of its unique principal curvature (which is no longer than its mean curvature).

The simplest and most obvious way of extending the last property to a k -umbilical submanifold is by requiring that it have the same principal curvatures with respect to all normal directions. A k -umbilical isometric immersion $f : M^n \rightarrow \tilde{M}^m$ is said to be *k-unipotent* if its principal curvatures $\kappa_i : UN_f^{m-1} \rightarrow \mathbb{R}$, $1 \leq i \leq k$, are constant. This is a very special symmetry condition that imposes strong geometric restrictions on f even punctually, since it will then shape up the same way in all directions. Prototypical examples of unipotent submanifolds of a Riemannian manifold \tilde{M}^m are given by the so-called normal homogeneous submanifolds, that is, orbits of a connected closed subgroup of $I^0(\tilde{M})$ whose slice representation at a point acts transitively on the unit sphere in the normal space at that point.

Remark 1. In [6], the first author used the notion of unipotency in the sense that the shape operator A_ξ satisfies the equation $A_\xi^2 = k \|\xi\|^2 I$, where $k > 0$ is constant, which except for hypersurfaces is easily seen to be equivalent to our definition of a 2-unipotent submanifold.

The second possibility is to require that the principal curvatures be constant just along parallel normal vector fields. Since these vector fields do not exist even locally in general, we must pick them always along piecewise differentiable curves, and this brings us to the classical concepts of submanifolds with constant principal curvatures and isoparametric submanifolds. Recall that an isometric immersion $f : M^n \rightarrow \tilde{M}^m$ is said to have constant principal curvatures if for any parallel normal vector field $\xi(t)$, along any piecewise differentiable curve, the principal curvatures in direction $\xi(t)$ are constant. If in addition the normal bundle of f is flat, one says that the submanifold is isoparametric.

Remark 2. Isoparametric hypersurfaces were studied by many prominent mathematicians in the 20th century, including B. Segre, Levi-Civita and É. Cartan, and they are still an active research field. The beginning of the history of their generalizations to higher codimensions goes back to the 80s, with the (sometimes independent) work of many authors: J. Eells [7], D. Gromoll and K. Grove [8], Q. M. Wang [9], C. E. Harle [10], W. Strübing [11] and C.-L. Terng [12]. There are different aspects of these generalizations that are actually strictly related: isoparametric maps, isoparametric submanifolds and submanifolds with constant principal curvatures (for more details on the historical development see [13]). The general notion of an isoparametric map is credited to C.-L. Terng. It turns out that regular level submanifolds of an isoparametric map are isoparametric submanifolds. Conversely, any isoparametric submanifold M^n of \mathbb{R}^m determines a polynomial isoparametric map on \mathbb{R}^m , which has M^n as a regular level set (cf. [14] and [2]). The notion of

isoparametric submanifold is nowadays also regarded as originally given by C.-L. Terng, even though it was first stated by C. E. Harle in [10]. There is a strong link between those submanifolds and s -representations. In fact, as a consequence of a result by Thorbergsson [15], the orbits of s -representations are almost all submanifolds with constant principal curvatures. It was conjectured by C. Olmos [16], twenty years ago, that a full and irreducible homogeneous submanifold of the sphere such that the normal holonomy group is not transitive must be an orbit of an irreducible s -representation. The conjecture was already known to be true for surfaces, and recently C. Olmos and R. Riaño-Riaño [17] have proved it also in dimension three. For many other interesting facts and results on isoparametric and homogeneous submanifolds, the interested reader can refer to Heintze-Olmos-Thorbergsson [18] and Berndt-Console-Olmos [2].

Since the condition of constant principal curvatures is far more important than having flat normal bundle and actually captures the essence of an isoparametric submanifold, we will make no distinction between them and submanifolds with constant principal curvatures, and both will be called CPC submanifolds for simplicity, whether their normal bundle is flat or not. It is also useful to indicate explicitly the constant number k of principal curvatures of f by saying that f is a k -CPC submanifold. Observe that, if $(x(t), \xi(t)) \in UN_f^{m-1}$ is a parallel normal vector field, then equation (1.3) yields

$$(x, \xi)' = (x', -f_* A_\xi x'),$$

and thus the constancy of the principal curvatures κ_i , $1 \leq i \leq k$, along (x, ξ) is equivalent to

$$\kappa_{i*}(x', -f_* A_\xi x') = 0.$$

In other words, f is k -CPC if and only if every κ_i is constant along the horizontal lift

$$T_{(x, \xi)}M = \{(X, -f_* A_\xi X) : X \in T_x M\}$$

of the tangent bundle.

Remark 3. *Unlike our definition of a CPC submanifold, those found in the literature do not require the submanifold to be k -umbilical.*

Finally, a last possible extension of the concept of an extrinsic sphere to k -umbilical submanifolds is obtained by requiring the even weaker condition that each principal curvature κ_i , $1 \leq i \leq k$, be constant but only along its lines of curvature, i.e., parallel normal vector fields $(x(t), \xi(t)) \in UN_f^{m-1}$ such that

$$A_\xi x' = \kappa_i x',$$

or equivalently, that each κ_i be constant along the horizontal lift

$$T_i = \{(X, -f_* A_\xi X) : X \in E_i\}$$

of its principal distribution E_i . In codimension one, T_i can be obviously identified with E_i and the above notion reduces to the definition of a Dupin hypersurface. It is hence quite natural to call a general k -umbilical isometric immersion $f : M^n \rightarrow \tilde{M}^m$ a k -Dupin submanifold if f satisfies the above property.

Remark 4. See Reckziegel [19] for a general Riemannian treatment of the notion of principal curvatures and curvature submanifolds in the case of an isometric immersion $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ of codimension greater than one. In that case, Reckziegel defines a curvature submanifold to be a connected submanifold $S \subset M^n$ for which there is a parallel (with respect to the normal connection) section of the unit normal bundle $\eta : S \rightarrow UN_f^{n+p-1}$ such that for each $x \in S$, the tangent space $T_x S$ is equal to some eigenspace of $A_{\eta(x)}$. The corresponding principal curvature $\kappa : S \rightarrow \mathbb{R}$ is then a smooth function on S . Pinkall [20] calls an isometric immersion $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ of codimension greater than one “Dupin” if all of its principal curvatures are Dupin, i.e., constant along each corresponding curvature submanifold (in the sense of Reckziegel). Pinkall’s definition is clearly equivalent to our definition of a k -Dupin submanifold in the case where f is k -umbilical. Furthermore, the notion of Dupin can be generalized even to the larger class of Legendre submanifolds, so that our definition of a Dupin submanifold can be expressed in terms of its Legendre lift, as we will see in Subsection 2.2. The basic idea is that a submanifold of codimension greater than one is k -Dupin if and only if its Legendre lift is $(k+1)$ -Dupin in the sense of Pinkall [21]. In any case, we are surprised that the concept of a Dupin submanifold of higher codimension has not yet been much explored in the literature. On the other hand, we mention, for the record, the work [22] by Dajczer, Florit and Tojeiro, in which they studied a weak notion of reducibility for submanifolds that carry a so-called Dupin principal normal.

The link with Lie sphere geometry provided by Legendre lifts is at the core of our approach throughout this article and allows us to deduce several interesting properties of Dupin submanifolds from the corresponding ones satisfied by their lifts. It will follow for instance that Dupin submanifolds are invariant under conformal transformations of and between ambient spaces, and also that the constancy of a principal curvature along the leaves of its principal foliation is automatic when its multiplicity is greater than one.

Remark 5. Unlike our definition of a Dupin submanifold, those found in the literature for hypersurfaces also do not require the submanifold to be k -umbilical, and the term “proper” is reserved for when this is the case.

It is clear from the above definitions that every k -unipotent submanifold is k -CPC and that every k -CPC submanifold is in turn k -Dupin, so that (1.4) holds in the obvious notation. It is also clear that for $k = 1$ the first notion only makes sense for hypersurfaces, in which case all three definitions collapse into that of an extrinsic sphere. Actually, in the case of hypersurfaces the first two concepts of k -unipotent and k -CPC hypersurfaces always coincide, for all $k \in \mathbb{N}$.

In the next case $k = 2$, the class of 2-unipotent submanifolds in space forms has been studied by the first author in [6], who has shown that in codimension greater than one it is nonempty essentially only in the spheres $\mathbb{S}^4, \mathbb{S}^7, \mathbb{S}^{13}, \mathbb{S}^{25}$ and in each case consists of the sole standard embedding of the real, complex, quaternionic or octonionic projective plane, respectively. The reason for the ‘essentially’ qualifier is that such embeddings can of course also be composed with the inverse of the similarity $\theta_c : \mathbb{S}_c^m \rightarrow \mathbb{S}^m$ given by

$$\theta_c(x) = \sqrt{c}x,$$

where $\mathbb{S}_c^m \subset \mathbb{R}^{m+1}$ is regarded as

$$\mathbb{S}_c^m = \left\{ x \in \mathbb{R}^{m+1} : \|x\|^2 = \frac{1}{c} \right\},$$

in order to yield the versions of the standard embeddings in $\mathbb{S}_c^4, \mathbb{S}_c^7, \mathbb{S}_c^{13}, \mathbb{S}_c^{25}$ instead.

Theorem 1 (Di Scala [6]). *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$, $p \geq 2$, be an isometric immersion. Then, the following facts are equivalent:*

- (i) *f is 2-unipotent;*
- (ii) *$c > 0$, $n = 2^\mu$, $p = 2^{\mu-1} + 1$ for $\mu = 1, 2, 3, 4$ and, up to congruence and the similarity θ_c , $f(M)$ is an open subset of $\psi_{\mathbb{F}}(\mathbb{FP}^2)$, where $\psi_{\mathbb{F}} : \mathbb{FP}^2 \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ denotes the standard embedding of the projective plane \mathbb{FP}^2 , for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, according to whether $\mu = 1, 2, 3, 4$, respectively.*

The standard embeddings of the projective planes are also familiar to us from Cartan's [23] classification of 3-isoparametric hypersurfaces, which together with the above remark on Legendre lifts suggests that they might exhaust the class of 2-CPC submanifolds in space forms too in codimension greater than one. This is actually the case, and is the content of one of our main results in this article.

Theorem 2. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$, $p \geq 2$, be a connected 2-CPC isometric immersion. Then $c > 0$, $n = 2^\mu$, $p = 2^{\mu-1} + 1$ for $\mu = 1, 2, 3, 4$ and there exists an isometry ϕ of $\mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1}$ such that $\theta_c \circ \phi \circ f(M)$ is an open subset of the image of the standard embedding $\psi_{\mathbb{F}} : \mathbb{FP}^2 \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ of the projective plane \mathbb{FP}^2 , where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively, and $\theta_c : \mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1} \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ is the similarity.*

Conversely, for every isometry ϕ of $\mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1}$, the composition $\phi \circ \theta_c^{-1} \circ \psi_{\mathbb{F}} : \mathbb{FP}^2 \rightarrow \mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1}$ is a 2-CPC isometric immersion.

Remark 6. *The remaining case $p = 1$ in Theorems 1 and 2 corresponds to the well-known classification of 2-isoparametric hypersurfaces in space forms. For \mathbb{R}^{n+1} this classification is due to Levi-Civita [24] for $n = 2$ and to Segre [25] in general, while for \mathbb{H}^{n+1} and \mathbb{S}^{n+1} it is due to E. Cartan [26], [23].*

A 2-isoparametric hypersurface of \mathbb{R}^{n+1} or \mathbb{H}^{n+1} is always a spherical cylinder (or tube around a totally geodesic submanifold, depending on the reader's philosophical bent), whereas in \mathbb{S}^{n+1} it is a Riemannian product of two spheres.

Remark 7. *For another interesting characterization of the Veronese submanifolds $\psi_{\mathbb{F}}$ see the recent paper [17] by C. Olmos and R. Riaño-Riaño. These submanifolds also appear implicitly or explicitly in several other contexts and works, such as in Console-Olmos [27], Nurowski [28], [29] and also [30] by the second author, just to give a few references. The importance of the Veronese embeddings in the geometry of submanifolds cannot be sufficiently emphasized. It seems that time and again, when people least expect it, new important developments come out involving these simple and compellingly beautiful submanifolds.*

Finally, concerning the last class of 2-Dupin submanifolds, in view of conformal invariance, many other spherical examples of these are obtained after a conformal deformation of one of the standard embeddings $\psi_{\mathbb{F}}$ and even in Euclidean and hyperbolic space forms via the stereographic projections $\pi_0 : \mathbb{R}^m \rightarrow \mathbb{S}^m$ and $\pi_c :$

$\mathbb{H}_c^m \rightarrow \mathbb{S}^m$ with pole $\sigma = (-1, 0, \dots, 0)$ given by

$$(1.5) \quad \begin{aligned} \pi_0(x) &= \left(\frac{1 - x \cdot x}{1 + x \cdot x}, \frac{2x}{1 + x \cdot x} \right), \\ \pi_c(x) &= \left(\frac{1 - y \cdot y}{1 + y \cdot y}, \frac{2y}{1 + y \cdot y} \right), \quad y = (x_2, \dots, x_{m+1}) / (x_1 + c^{-1/2}), \end{aligned}$$

where we regard \mathbb{H}_c^m in the Lorentzian space \mathbb{L}^{m+1} as

$$\mathbb{H}_c^m = \left\{ x \in \mathbb{L}^{m+1} : \|x\|^2 = \frac{1}{c} \right\}.$$

Nevertheless, the main goal of this paper is to prove that these are the only possible new examples of 2-Dupin submanifolds in space forms. Since these submanifolds are invariant under conformal transformations between ambient spaces, the statement is essentially the same whether it is considered in the sphere, in Euclidean space or in hyperbolic space. Among these three geometries, the spherical one is the best suited for displaying solutions of problems in conformal geometry with some Riemannian-geometric flavor, especially for the fact that the sphere has this distinguished property that any two given extrinsic spheres of the same dimension are always similar, while in the other two space forms also stereographic projections must be taken into account in order to move from one type of extrinsic sphere to another. Hence, we confine ourselves here to stating our classification of 2-Dupin submanifolds just in the unit sphere \mathbb{S}^m .

Theorem 3. *Let $f : M^n \rightarrow \mathbb{S}^{n+p}$, $p \geq 2$, be a connected 2-Dupin isometric immersion. Then $n = 2^\mu$, $p = 2^{\mu-1} + 1$ for $\mu = 1, 2, 3, 4$ and there exists a conformal transformation ϕ of $\mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ such that $\phi \circ f(M)$ is an open subset of the image of the standard embedding $\psi_{\mathbb{F}} : \mathbb{FP}^2 \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ of the projective plane \mathbb{FP}^2 , where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively.*

Conversely, for every conformal transformation ϕ of $\mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$, the composition $\phi \circ \psi_{\mathbb{F}}$ is a 2-Dupin isometric immersion.

Remark 8. *The remaining case $p = 1$ of hypersurfaces in the above theorem corresponds to Pinkall's [21] celebrated characterization of the well-known cyclides of Dupin as being always locally conformally congruent to rotational hypersurfaces with a sphere as profile.*

The proof of Theorem 3 relies on Pinkall [31] and Cecil-Jensen's [32] deep result that an irreducible 3-Dupin hypersurface must be Lie equivalent to a 3-isoparametric hypersurface, which in turn is a tube over the standard embedding $\psi_{\mathbb{F}}$ of the projective plane \mathbb{FP}^2 , for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, in $\mathbb{S}^4, \mathbb{S}^7, \mathbb{S}^{13}, \mathbb{S}^{25}$, respectively, by Cartan's classification. Given a 2-Dupin submanifold $f : M^n \rightarrow \mathbb{S}^{n+p}$, $p \geq 2$, our proof can be briefly outlined as follows.

- (1) We consider the Legendre lift $\lambda : UN_f^{n+p-1} \rightarrow \Lambda^{2(n+p)-1}$ of f , which will be a 3-Dupin submanifold in the sense of Lie sphere geometry.
- (2) We manage to prove that λ is irreducible in the sense of Pinkall [21].
- (3) We then use Pinkall and Cecil-Jensen's aforementioned result coupled with Cartan's classification of 3-isoparametric hypersurfaces to conclude that λ is Lie equivalent to a tube over some $\psi_{\mathbb{F}}$.

- (4) We use a characterization of Lie sphere transformations by Cecil-Chern [33] according to which any such transformation can be written as the composition of two Möbius transformations and some parallel transformation in the middle.
- (5) We finally show that if the above parallel transformation existed, then f would be 2-unipotent, and therefore it would already be itself one of the standard embeddings $\psi_{\mathbb{F}}$, by Theorem 1.

We now turn our attention to the broader class of 2-umbilical submanifolds of space forms. The fact mentioned above that the only principal curvatures which must be considered in checking the Dupin property are those of multiplicity one takes us one step away from the complete classification of these submanifolds too. In fact, as we will see in the next section, the two distinct principal curvatures of a 2-umbilical submanifold $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ of codimension $p > 1$ have always the same multiplicity m , and in particular M^n has even dimension $n = 2m$. Therefore, every 2-umbilical submanifold that is not a surface nor a hypersurface is automatically 2-Dupin, and we are covered by Theorem 3. In the case of a surface $f : M^2 \rightarrow \mathbb{Q}_c^{2+p}$, there is not much to say, except that $p \leq 2$, since the vector subspace of real symmetric 2×2 matrices is 3-dimensional and f has no umbilical normal directions by definition. Thus, the only possibility remaining is that $f : M^n \rightarrow \mathbb{Q}_c^{n+1}$ is a hypersurface with a principal curvature κ of multiplicity $n - 1 \geq 2$. These are the well-known generalized $(n - 1)$ -cylinders for $\kappa = 0$ and 1-parameter envelopes of hyperspheres (briefly, 1-PES) otherwise, which were studied by Dajczer-Florit-Tojeiro [22] and Asperti-Dajczer [34], respectively. For the sake of completeness, we include a brief account of these results in Subsection 2.3, where we also offer an erratum to one of Asperti-Dajczer's results about the so-called special k -parameter envelopes of spheres (briefly, k -SPES).

Coupling Theorem 3 with the above considerations, we get the following complete classification of 2-umbilical submanifolds of space forms. Since the k -umbilical property is also invariant under conformal transformations between ambient spaces (because the traceless parts of the shape operators do not change under such a transformation), we present also this result just in the sphere.

Theorem 4. *Let $f : M^n \rightarrow \mathbb{S}^{n+p}$ be a 2-umbilical isometric immersion. Then, one of the following possibilities holds:*

- (i) $n = 2^\mu$, $p = 2^{\mu-1} + 1$ for $\mu = 2, 3, 4$ and there exists a conformal transformation ϕ of $\mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ such that $\phi \circ f(M)$ is an open subset of the image of the standard embedding $\psi_{\mathbb{F}} : \mathbb{FP}^2 \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ of the projective plane \mathbb{FP}^2 , where $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively.
- (ii) $n \geq 3$, $p = 1$ and f is locally conformally congruent to a rotational hypersurface with a sphere as profile.
- (iii) $n \geq 3$, $p = 1$ and, on each connected component of an open dense subset of M^n , f is either a generalized $(n - 1)$ -cylinder or a 1-PES.
- (iv) $n = 2$, $p \leq 2$ and f is a surface free of umbilical normal directions.

Finally, we also study a weaker notion of 2-umbilicality that allows the existence of umbilical normal directions. An isometric immersion $f : M^n \rightarrow \tilde{M}^m$ is said to be *weakly k -umbilical* at $x \in M^n$ if it has at most k distinct principal curvatures with respect to all $(x, \xi) \in UN_f^{m-1}$. A submanifold is called *weakly k -umbilical* if it is weakly k -umbilical at every point. In this case the principal curvatures

$\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n$ of f with respect to (x, ξ) are still continuous in the whole UN_f^{m-1} , but since they no longer have constant multiplicity, smoothness can be now ensured only on a open dense subset of UN_f^{m-1} . Note also that with this definition every weakly $(k-1)$ -umbilical submanifold is also weakly k -umbilical. In particular, umbilical submanifolds are weakly 2-umbilical.

Remark 9. *The above idea of weak k -umbilicality should not be confused with the concept of weak-umbilic points introduced by Moore [35].*

One might hope that this new concept gives rise to some other interesting types of submanifolds in space forms in the case $k = 2$. However, we break this expectation by showing that the new possibilities that come out of this definition are not much distinct from the examples we have already gotten in Theorem 4. Before stating the result, first note that the standard embeddings in case (i) of Theorem 4 have flat normal bundle nowhere, while the rotational hypersurfaces, generalized cylinders and envelopes of hyperspheres in cases (ii) and (iii), being hypersurfaces, have flat normal bundle everywhere. It turns out that this duality is preserved in the weakly 2-umbilical case. We show that allowing umbilical normal directions keeps cases (i) and (ii) essentially unaffected, as they remain true up to reduction of conformal codimension. Regarding case (iii), the notion of a generalized cylinder actually makes sense in arbitrary codimension, while 1-PES are replaced by their natural generalization in higher codimension described in Subsection 2.3. Lastly, in case (iv) all surfaces of arbitrary codimension are now clearly allowed. To avoid technicalities we shall assume f in the statement to be umbilic free.

Theorem 5. *Let $f : M^n \rightarrow \mathbb{S}^{n+p}$ be an umbilic-free weakly 2-umbilical isometric immersion. Then, one of the following possibilities holds:*

- (i) $R^\perp \neq 0$ everywhere, $n = 2^\mu$ for $\mu = 2, 3, 4$ and there exist an umbilical sphere $\mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1}$, $c \geq 1$, of $\mathbb{S}^{3 \cdot 2^\mu + p}$ such that $f(M) \subset \mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1}$ and a conformal transformation ϕ of $\mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1}$ such that $\theta_c^{-1} \circ \phi \circ f(M)$ is an open subset of the image of the standard embedding $\psi_{\mathbb{F}} : \mathbb{FP}^2 \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ of the projective plane \mathbb{FP}^2 , where $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively, and $\theta_c : \mathbb{S}_c^{3 \cdot 2^{\mu-1} + 1} \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ is the similarity.
- (ii) $R^\perp = 0$ everywhere, $n \geq 3$ and there exists an umbilical sphere \mathbb{S}_c^{n+1} , $c \geq 1$, of \mathbb{S}^{n+p} such that $f(M) \subset \mathbb{S}_c^{n+1}$ and f is locally conformally congruent to a rotational hypersurface in \mathbb{S}_c^{n+1} with a sphere as profile.
- (iii) $R^\perp = 0$ everywhere, $n \geq 3$ and, on each connected component of an open dense subset of M^n , f either is a generalized $(n-1)$ -cylinder or envelopes a 1-parameter congruence of hyperspheres.
- (iv) $n = 2$ and f is an umbilic-free surface.

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The second author is also looking forward to Rio 2016 and he hopes to see everybody there!

2. PRELIMINARIES

Before proving our main theorems, we begin with some preliminary results on 2-umbilical submanifolds, Dupin submanifolds and 1-PES. To facilitate the exposition, we will divide this section into three subsections. The results in the first

subsection derive from purely algebraic considerations on symmetric bilinear forms mirroring second fundamental forms of 2-umbilical submanifolds, while the second subsection consists of well-known results from Lie sphere geometry and its applications to the theory of Dupin submanifolds. The last subsection is devoted to the study of generalized $(n - 1)$ -cylinders and 1-PES and their generalization to higher codimension.

2.1. 2-umbilical submanifolds. In this subsection, we derive two important results concerning 2-umbilical submanifolds. The first of these provides a simple (necessary and) sufficient condition for a 2-umbilical submanifold to be 2-unipotent and the second describes the spectral structure of the second fundamental form of 2-umbilical submanifolds.

In order to state the above lemmas we need to introduce some terminology. Let V and W be real vector spaces of finite dimension with positive definite inner products and let $\beta : V \times V \rightarrow W$ be a symmetric bilinear form. For any given $\xi \in W$, let $B_\xi : V \rightarrow V$ be the self-adjoint operator defined by

$$\langle B_\xi X, Y \rangle = \langle \beta(X, Y), \xi \rangle.$$

Motivated by the situation of the second fundamental form of k -umbilical submanifolds, β is said to be *k-umbilical* if B_ξ has exactly k distinct eigenvalues $\kappa_i(\xi)$, $1 \leq i \leq k$, for all $\xi \in W$ with $\|\xi\| = 1$. It is called *k-unipotent* if in addition either $k = 1$ or $k \geq 2$ and the eigenvalues $\kappa_i(\xi) = \kappa_i$, $1 \leq i \leq k$, are constant.

In our target case of 2-umbilical symmetric bilinear forms, the next lemma shows that it suffices to check that all B_ξ 's share just one eigenvalue for β to be 2-unipotent.

Lemma. *Let $\beta : V^n \times V^n \rightarrow W^p$ be a 2-umbilical symmetric bilinear form. If all B_ξ , $\|\xi\| = 1$, share a common eigenvalue κ , then β is 2-unipotent.*

Proof. In the case $p = 1$ there is nothing to prove. Otherwise, we first claim that $\kappa \neq 0$. To see this, consider the vector $\mathcal{H} \in W^p$ defined by

$$\mathcal{H} = \frac{1}{n} \sum_{j=1}^n \beta(X_j, X_j)$$

in terms of an orthonormal basis X_1, \dots, X_n of V^n .

The preceding expression implies that

$$n \langle \mathcal{H}, \xi \rangle = \text{tr } B_\xi$$

for any $\xi \in W^p$, hence

$$\text{tr } B_\xi = 0$$

for any unit vector $\xi \in W^p$ orthogonal to \mathcal{H} . Since B_ξ has exactly two distinct eigenvalues, the above equation gives

$$\kappa \neq 0,$$

as we wished. Then, since by assumption κ must be a common eigenvalue of both B_ξ and $B_{-\xi} = -B_\xi$, it follows that $\pm\kappa$ are their two eigenvalues. \square

The above lemma has the following geometric counterpart.

Lemma 1. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ be a 2-umbilical isometric immersion. If all shape operators A_ξ , $(x, \xi) \in UN_f^{n+p-1}$, share a common principal curvature κ , then f is 2-unipotent.*

Proof. This is an immediate consequence of the previous lemma for $\beta = \alpha_x$, $x \in M^n$. \square

Next, we show that 2-umbilicality imposes a very special spectral structure for symmetric bilinear forms. In particular, it will follow that, in codimension $p \geq 2$, this phenomenon is unique to even dimensions.

Lemma. *Let $\beta : V^n \times V^n \rightarrow W^p$, $p \geq 2$, be a 2-umbilical symmetric bilinear form. Then, for every nonzero vector $\xi \in W^p$, we have*

$$(2.1) \quad \kappa_i(-\xi) = -\kappa_j(\xi), \quad 1 \leq i \neq j \leq 2,$$

and thus

$$(2.2) \quad m_1(\xi) = m_2(\xi) =: m.$$

In particular, $n = 2m$ is even.

Proof. Since B_ξ has always two distinct eigenvalues, so does its traceless part B_ξ^0 , whose eigenvalues are then $\pm \kappa_0(\xi) \neq 0$. As $p \geq 2$, we can walk from ξ to $-\xi$ without stepping foot at the origin, so κ_0 never vanishes on the way and hence

$$\kappa_0(-\xi) = \kappa_0(\xi),$$

which easily yields (2.1). Moreover, the fact that $E_{-\kappa_j}(-\xi) = E_{\kappa_j}(\xi)$, together with (2.1), clearly implies (2.2). \square

Again we state the immediate geometric counterpart of the previous lemma.

Lemma 2. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$, $p \geq 2$, be a 2-umbilical isometric immersion. Then, for every $(x, \xi) \in UN_f^{n+p-1}$, we have*

$$\kappa_i(x, -\xi) = -\kappa_j(x, \xi), \quad 1 \leq i \neq j \leq 2,$$

and thus

$$m_1(x, \xi) = m_2(x, \xi) =: m.$$

In particular, $n = 2m$ is even.

Proof. Again a direct consequence of the preceding lemma for $\beta = \alpha_x$, $x \in M^n$. \square

Remark 10. *By relatively simple similar arguments the statements of Lemmas 1 and 2 can be extended to weakly 2-umbilical submanifolds:*

- (i) *The corresponding version of Lemma 1 assumes that f is umbilic-free and the shape operators A_ξ share a common principal curvature κ for all nonumbilical normal directions $(x, \xi) \in UN_f^{n+p-1}$. The conclusion is that either $\kappa = 0$ and the first normal space $N_1^f(x) = \text{span}\{\mathcal{H}(x)\}$ has dimension 1 for all $x \in M^n$, in particular, the second fundamental form of f is diagonalizable and f has flat normal bundle everywhere by the Ricci equation, or $\kappa \neq 0$ and f is 2-unipotent;*
- (ii) *The assertions in Lemma 2 remain true for every nonumbilical normal direction $(x, \xi) \in UN_f^{n+p-1}$ at every point $x \in M^n$ such that $R^\perp(x) \neq 0$ in the weakly 2-umbilical case, in particular, $n = 2m$ is even if f has nonflat normal bundle somewhere.*

2.2. Lie sphere geometry and Dupin submanifolds. Lie [36] introduced the geometry of oriented spheres in his dissertation, published as a paper in *Mathematische Annalen* in 1872. Sphere geometry was also prominent in his study of contact transformations (Lie-Scheffers [37]) and in Volume III of Blaschke's book on differential geometry published in 1929.

Lie sphere geometry has become a valuable tool in the study of Dupin submanifolds, beginning with Pinkall's [38] dissertation in 1981. In this subsection, we will present a series of results that will illustrate this value. The first result is an important characterization of the Lie sphere group in terms of Möbius and parallel transformations due to Cecil and Chern [33]. The next couple of results provide the basic link between Dupin submanifolds as defined in the introduction of this article and those in the context of Legendre submanifolds by means of the so-called Legendre lifts. This connection will allow us to deduce several properties of our Dupin submanifolds out of the ones satisfied by their Legendre lifts. It will follow, for example, that the only principal curvatures which must be considered in checking the Dupin property are those of multiplicity one, and also that our class of Dupin submanifolds is preserved by conformal transformations of and between ambient space forms.

Finally, we will discuss the classification of proper Dupin submanifolds with three curvature spheres by Cecil and Jensen [32]. A crucial assumption in their result is the irreducibility of the Dupin submanifold in the sense of Pinkall [21]. In the next section, we will manage to prove that the Legendre lifts arising from our Dupin submanifolds always satisfy such condition, and for this purpose stating Pinkall's [38] simple Lie sphere geometric criterion for reducibility will come in handy.

Before and during our presentation of the above results, we will be briefly reviewing some important concepts from Lie sphere geometry as we need them. A detailed account on the subject can be found in [39].

Start with the real vector space \mathbb{R}_2^{d+3} endowed with the inner product of signature $(-1, 1, \dots, 1, -1)$ and define an equivalence relation on $\mathbb{R}_2^{d+3} - \{0\}$ by setting $x \simeq y$ if $x = ty$ for some nonzero real number t . We denote the equivalence class determined by a vector x by $[x]$. Projective space \mathbb{P}^{d+2} is the set of such equivalence classes, and it can naturally be identified with the space of all lines through the origin in \mathbb{R}_2^{d+3} . Let Q^{d+1} be the quadric in \mathbb{P}^{d+2} given in homogeneous coordinates by the equation

$$(2.3) \quad \langle x, x \rangle = -x_1^2 + x_2^2 + \dots + x_{d+2}^2 - x_{d+3}^2 = 0.$$

The manifold Q^{d+1} is called the *Lie quadric*, and the scalar product determined by the quadratic form in (2.3) is called the *Lie metric* or *Lie scalar product*. We will let $\{e_1, \dots, e_{d+3}\}$ denote the standard orthonormal basis for \mathbb{R}_2^{d+3} . It turns out that points on Q^{d+1} correspond to the set of oriented hyperspheres and point spheres (hyperspheres with radius 0) in \mathbb{S}_c^d , or in general to the set of oriented extrinsic hyperspheres and point spheres in $\mathbb{Q}_c^d \cup \{\infty\}$ for the Euclidean or a hyperbolic space form \mathbb{Q}_c^d , $c \leq 0$, where the *improper point* ∞ is identified with the pole σ of the stereographic projection π_c given in equation (1.5). Actually, from the point of view of Klein's Erlangen Program, all three spherical, Euclidean and hyperbolic geometries are subgeometries of Lie sphere geometry (see [39] for the details). On the other hand, in some ways it is simpler to use the sphere \mathbb{S}^d rather than \mathbb{R}^d or \mathbb{H}^d as the base space for the study of Möbius or Lie sphere geometry, since this avoids the use of stereographic projection and the need to refer to an improper

point or to distinguish between spheres, planes and horospheres. Indeed, the above correspondence between points in Q^{d+1} and oriented spheres in the base space is much simpler in \mathbb{S}^d than in \mathbb{R}^d and \mathbb{H}^d . It is given simply by

$$(2.4) \quad S(x, r) \leftrightarrow [(\cos r, x, \sin r)],$$

where $S(x, r)$ denotes the sphere in \mathbb{S}^d with center x and signed radius r . In particular, the point sphere $x = S(x, 0)$ in \mathbb{S}^d corresponds to the point $[(1, x, 0)]$ in Q^{d+1} . It is an important fact that the Lie quadric contains projective lines but no linear subspaces of higher dimension.

The set of oriented spheres in \mathbb{S}^d corresponding to the points on a line on Q^{d+1} forms a so-called *parabolic pencil* of spheres. Each parabolic pencil contains exactly one point sphere k_1 and one great sphere k_2 . The pencil consists of all oriented hyperspheres in oriented contact with k_1 and k_2 .

A *Lie sphere transformation* is a projective transformation of \mathbb{P}^{d+2} which takes Q^{d+1} to itself. In terms of the geometry of \mathbb{S}^d , a Lie sphere transformation maps Lie spheres to Lie spheres. (Here the term “Lie sphere” includes oriented spheres and point spheres.) Furthermore, since it is projective, a Lie sphere transformation maps lines on Q^{d+1} to lines on Q^{d+1} . Thus, it preserves oriented contact of spheres in \mathbb{S}^d . The group G of Lie sphere transformations is isomorphic to $O(d+1, 2) / \{\pm I\}$, where $O(d+1, 2)$ is the group of orthogonal transformations of \mathbb{R}_2^{d+3} . Pinkall’s [40]–[41] so-called “fundamental theorem of Lie sphere geometry” states that any line preserving diffeomorphism of Q^{d+1} is the restriction to Q^{d+1} of a projective transformation, that is, a transformation of the space of oriented spheres which preserves oriented contact must be a Lie sphere transformation.

Recall that a linear transformation $L \in GL(d+1)$ induces a projective transformation $P(L)$ on \mathbb{P}^d defined by $P(L)[x] = [Lx]$. The map P is a homomorphism of $GL(d+1)$ onto the group $PGL(d)$ of projective transformations of \mathbb{P}^d . It is well known (see, for example, Samuel [42]) that the kernel of P is the group of all nonzero scalar multiples of the identity transformation I . A Möbius transformation is a transformation on the space of unoriented spheres, i.e., the space of projective classes of spacelike vectors in \mathbb{R}_1^{d+2} . Such a transformation also takes lightlike vectors to lightlike vectors, and so it induces a conformal diffeomorphism of the sphere \mathbb{S}^d (regarded as the Möbius quadric in \mathbb{P}^{d+1}) onto itself.

It is well known that the group of conformal diffeomorphisms of the sphere is precisely the Möbius group. Now, each Möbius transformation naturally induces two Lie sphere transformations on the space Q^{d+1} of oriented spheres. Specifically, if L is in $O(d+1, 1)$, then we can extend L to a transformation \bar{L} in $O(d+1, 2)$ by setting $\bar{L} = L$ on \mathbb{R}_1^{d+2} and $\bar{L}(e_{d+3}) = \pm e_{d+3}$, where the sign determines the orientation of every oriented sphere. Thus, the group of Lie transformations induced from Möbius transformations consists of those Lie transformations that map $[e_{d+3}]$ to itself. Since such a transformation must also take e_{d+3}^\perp to itself, this is precisely the group of Lie transformations which take point spheres to point spheres.

When working in the context of Lie sphere geometry, it is a common abuse of language to refer to these transformations also as “Möbius transformations”. Besides Möbius transformations, an important family of Lie transformations is given by the so-called parallel transformations P_t , which add t to the signed radius of each sphere while keeping the center fixed. For each of the spherical, Euclidean and hyperbolic geometries, there is a corresponding one-parameter family of parallel transformations. *Spherical parallel transformation* P_t is accomplished by the

following transformation in $O(d+1, 2)$:

$$(2.5) \quad \begin{aligned} P_t e_1 &= \cos t e_1 + \sin t e_{d+3}, \\ P_t e_{d+3} &= -\sin t e_1 + \cos t e_{d+3}, \\ P_t e_i &= e_i, \quad 2 \leq i \leq d+2. \end{aligned}$$

On the other hand, *Euclidean parallel transformation* P_t is accomplished in terms of matrix representation with respect to the standard orthonormal basis by

$$(2.6) \quad P_t = \begin{bmatrix} 1 - (t^2/2) & -t^2/2 & 0 & -t \\ t^2/2 & 1 + (t^2/2) & 0 & t \\ 0 & 0 & I & 0 \\ t & t & 0 & 1 \end{bmatrix}.$$

Lastly, *hyperbolic parallel transformation* is accomplished by the following transformation:

$$(2.7) \quad \begin{aligned} P_t e_i &= e_i, \quad i = 1, 3, \dots, d+2, \\ P_t e_2 &= \cosh t e_2 + \sinh t e_{d+3}, \\ P_t e_{d+3} &= \sinh t e_2 + \cosh t e_{d+3}. \end{aligned}$$

The following theorem demonstrates the important role played by parallel transformations in generating the Lie sphere group (see Cecil-Chern [33]).

Theorem 6 (Cecil-Chern [33]). *Any Lie sphere transformation γ can be written as*

$$\gamma = \Phi_1 P_t \Phi_2,$$

where Φ_1 and Φ_2 are Möbius transformations and P_t is some Euclidean, spherical or hyperbolic parallel transformation.

Let us now develop the framework necessary to study submanifolds within the context of Lie sphere geometry. The manifold Λ^{2d-1} of projective lines on the Lie quadric Q^{d+1} can be identified with the tangent bundle $T_1 \mathbb{S}^d$ of \mathbb{S}^d and thus borrows from the latter a contact structure, i.e., a globally defined 1-form ω such that $\omega \wedge (d\omega)^{d-1} \neq 0$ on Λ^{2d-1} . This gives rise to a codimension one distribution D on Λ^{2d-1} that has integral submanifolds of dimension $d-1$, but none of higher dimension. These integral submanifolds are called Legendre submanifolds. Any submanifold of a real space-form \mathbb{S}^d , \mathbb{R}^d or \mathbb{H}^d naturally induces a Legendre submanifold, and thus Lie sphere geometry can be used to analyze submanifolds in these spaces. This has been particularly effective in the classification of Dupin submanifolds.

To see how a submanifold in a space form gives rise to a Legendre submanifold, first suppose that $f : M^n \rightarrow \mathbb{S}^{n+1}$ is an immersed oriented hypersurface with field of unit normals $\xi : M^n \rightarrow \mathbb{S}^{n+1}$. The induced Legendre submanifold is then given by the map $\lambda : M^n \rightarrow \Lambda^{2n+1}$ defined by

$$\lambda(x) = [Y_1(x), Y_{n+4}(x)],$$

where

$$Y_1(x) = (1, f(x), 0), \quad Y_{n+4}(x) = (0, \xi(x), 1).$$

The map λ is called the *Legendre lift* of the immersion f with field of unit normals ξ . Next, we handle the case of a submanifold $f : M^n \rightarrow \mathbb{S}^{n+p}$ of codimension p

greater than one. The *Legendre lift* of f is the map $\lambda : UN_f^{n+p-1} \rightarrow \Lambda^{2(n+p)-1}$ defined by

$$\lambda(x, \xi) = [Y_1(x), Y_{n+p+3}(x, \xi)],$$

where

$$(2.8) \quad Y_1(x) = (1, f(x), 0), \quad Y_{n+p+3}(x, \xi) = (0, \xi, 1).$$

Geometrically, $\lambda(x, \xi)$ is the line on the quadric Q^{n+p+1} corresponding to the parabolic pencil of spheres in \mathbb{S}^{n+p} in oriented contact at the contact element $(f(x), \xi) \in T_1\mathbb{S}^{n+p}$. The situation for submanifolds of \mathbb{R}^{n+p} or \mathbb{H}^{n+p} is similar and we shall not treat it here (cf. Cecil [39]). Now suppose that $\lambda : N^{d-1} \rightarrow \Lambda^{2d-1}$ is an arbitrary Legendre submanifold. Then, it is always possible to parametrize λ by the point sphere map $[Y_1]$ and the great sphere map $[Y_{d+3}]$ given by

$$(2.9) \quad Y_1 = (1, f, 0), \quad Y_{d+3} = (0, \xi, 1).$$

This defines two maps f and ξ from N^{d-1} to \mathbb{S}^d , which we call the *spherical projection* and *spherical field of unit normals*, respectively. Both f and ξ are smooth maps, but neither need to be an immersion or even have constant rank. The Legendre submanifold induced by an oriented hypersurface in \mathbb{S}^d is the special case where the spherical projection f is an immersion, i.e., f has constant rank $d-1$ on N^{d-1} . In the case of the Legendre submanifold induced by a submanifold $f : M^n \rightarrow \mathbb{S}^{n+p}$, the spherical projection $f : UN_f^{n+p-1} \rightarrow \mathbb{S}^{n+p}$ given by $f(x, \xi) = f(x)$ is a submersion onto the image of f , i.e., it has constant rank n . The notions of projection and field of unit normals also have their Euclidean and hyperbolic versions.

Remark 11. *The jump from 1 to $d+3$, $d = n+p$, in the indices of the two Y_i 's defining the Legendre lift may seem a bit messy, but the notation is traditional from the method of moving frames in the context of Lie sphere geometry, where Y_1 and Y_{d+3} are just the first and last vectors of an ordered set called a Lie frame (refer to Cecil-Chern [33]).*

We now discuss the definition of a curvature sphere of an isometric immersion $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$, which is closely related to focal points of f , that is, singular points of the tube $f_t : UN_f^{n+p-1} \rightarrow \mathbb{Q}_c^{n+p}$ of radius t around f given by

$$\begin{aligned} f_t(x, \xi) &= \cos(\sqrt{ct}) f(x) + \frac{\sin(\sqrt{ct})}{\sqrt{c}} \xi, \quad \text{if } c > 0, \\ f_t(x, \xi) &= f(x) + t\xi, \quad \text{if } c = 0, \\ f_t(x, \xi) &= \cosh(\sqrt{-ct}) f(x) + \frac{\sinh(\sqrt{-ct})}{\sqrt{-c}} \xi, \quad \text{if } c < 0. \end{aligned}$$

Note that if $f : M^n \rightarrow \mathbb{Q}_c^{n+1}$ is an oriented hypersurface with unit normal vector field ξ , then $UN_f^n = \{(x, \xi(x)) : x \in M^n\}$ and the above tube is essentially the *parallel hypersurface* $f_t : M^n \rightarrow \mathbb{Q}_c^{n+1}$ of f at distance t defined by

$$\begin{aligned} f_t(x) &= \cos(\sqrt{ct}) f(x) + \frac{\sin(\sqrt{ct})}{\sqrt{c}} \xi(x), \quad \text{if } c > 0, \\ f_t(x) &= f(x) + t\xi(x), \quad \text{if } c = 0, \\ f_t(x) &= \cosh(\sqrt{-ct}) f(x) + \frac{\sinh(\sqrt{-ct})}{\sqrt{-c}} \xi(x), \quad \text{if } c < 0. \end{aligned}$$

For simplicity we assume $c = 1$, so

$$(2.10) \quad f_t(x, \xi) = \cos t f(x) + \sin t \xi.$$

Geometrically, one thinks of focal points as points where nearby normal geodesics intersect. It is well known that the location of focal points is related to the principal curvatures. Specifically, if $(X, -f_*A_\xi X + \eta) \in T_{(x, \xi)}UN_f$, then we have

$$\begin{aligned} f_{t*}(X, -f_*A_\xi X + \eta) &= \cos t f_*X + \sin t (-f_*A_\xi X + \eta) \\ &= f_*(\cos t X - \sin t A_\xi X) + \sin t \eta. \end{aligned}$$

Thus, $f_{t*}(X, -f_*A_\xi X + \eta) = 0$ for $(X, -f_*A_\xi X + \eta) \neq 0$ if and only if either $p \geq 2$, $t = k\pi$, $k \in \mathbb{Z}$, and $X = 0$ or $t \neq k\pi$, $k \in \mathbb{Z}$, $\eta = 0$, $\cot t$ is a principal curvature of f at (x, ξ) and X is a corresponding principal vector. Hence, $f_{k\pi}(x, \xi) = (-1)^k f(x)$, $k \in \mathbb{Z}$, is a focal point of f at (x, ξ) of multiplicity $p - 1$ if $p \geq 2$ and $y = f_t(x, \xi)$ is a focal point of f at (x, ξ) of multiplicity m if and only if $\cot t$ is a principal curvature of multiplicity m at (x, ξ) . If $p \geq 2$, it is convenient to introduce an extra principal curvature $\kappa = \cot 0 = \infty$ of multiplicity $p - 1$ associated to the focal points $\pm f(x)$. Then each principal curvature

$$\kappa = \cot t, \quad 0 \leq t < \pi,$$

produces two distinct antipodal focal points on the normal geodesic with parameter values t and $t + \pi$. The oriented hypersphere centered at a focal point $y = f_t(x, \xi)$ and in oriented contact with $f(M)$ at $(f(x), \xi)$ is called a *curvature sphere* of f at (x, ξ) . The two antipodal focal points determined by κ are the two centers of the corresponding curvature sphere. Thus, the correspondence between principal curvatures and curvature spheres is bijective.

The *multiplicity* of the curvature sphere is by definition equal to the multiplicity of the corresponding principal curvature. The *principal space* corresponding to the curvature sphere is by definition equal to the horizontal lift of the principal space of the corresponding principal curvature $\kappa = \cot t$ for $0 < t < \pi$ or $\{(0, \eta) : \eta \in N_f M(x), \eta \cdot \xi = 0\}$ for $t = 0$. We now consider these ideas as they apply to the Legendre lift of the isometric immersion $f : M^n \rightarrow \mathbb{S}^{n+p}$. As in equation (2.8), we have $\lambda = [Y_1, Y_{n+p+3}]$, where

$$Y_1 = (1, f, 0), \quad Y_{n+p+3} = (0, \xi, 1).$$

For each $(x, \xi) \in UN_f^{n+p-1}$, the points on the line $\lambda(x, \xi)$ can be parametrized as

$$[K_t(x, \xi)] = [\cos t Y_1(x) + \sin t Y_{n+p+3}(x, \xi)] = [(\cos t, f_t(x, \xi), \sin t)],$$

where f_t is given in equation (2.10). By equation (2.4), the point $[K_t(x, \xi)]$ in Q^{n+p+1} corresponds to the oriented sphere in \mathbb{S}^{n+p} with center $f_t(x, \xi)$ and signed radius t . This sphere is in oriented contact with the submanifold $f(M)$ at $(f(x), \xi)$. Given a tangent vector $(X, -f_*A_\xi X + \eta) \in T_{(x, \xi)}UN_f$, we have

$$K_{t*}(X, -f_*A_\xi X + \eta) = (0, f_{t*}(X, -f_*A_\xi X + \eta), 0).$$

Thus, $K_{t*}(X, -f_*A_\xi X + \eta) = (0, 0, 0)$ if and only if $f_{t*}(X, -f_*A_\xi X + \eta) = 0$, i.e., $y = f_t(x, \xi)$ is a focal point of f at (x, ξ) . Hence, we have that the point $[K_t(x, \xi)]$ in Q^{n+p+1} corresponds to a curvature sphere of the isometric immersion f at (x, ξ) if and only if

$$(2.11) \quad K_{t*}(X, -f_*A_\xi X + \eta) = (0, 0, 0)$$

for some nonzero vector $(X, -f_*A_\xi X + \eta) \in T_{(x,\xi)}UN_f$. This viewpoint allows us to extend the definition of curvature spheres and k -umbilicality to an arbitrary Legendre submanifold in the obvious way.

Remark 12. *Note that if we use a different parametrization*

$$[\tilde{K}_t(x, \xi)] = [\rho(x, \xi) K_t(x, \xi)]$$

of $[K_t(x, \xi)]$, then condition (2.11) for \tilde{K}_t is that $\tilde{K}_{t*}(X, -f_*A_\xi X + \eta)$ is collinear with \tilde{K}_t .

One can show that the notion of curvature sphere is invariant under Lie sphere transformations. Let $\lambda : N^{d-1} \rightarrow \Lambda^{2d-1}$ be a Legendre submanifold parametrized by $\lambda = [Z_1, Z_{d+3}]$. Suppose $\gamma = P(L)$ is the Lie sphere transformation induced by an orthogonal transformation L in the group $O(d+1, 2)$. Since L is orthogonal, it is easy to check that the map defined by $[LZ_1, LZ_{d+3}]$ is also a Legendre submanifold, which we will denote by $\gamma\lambda : N^{d-1} \rightarrow \Lambda^{2d-1}$. The Legendre submanifolds λ and $\gamma\lambda$ are said to be *Lie equivalent*. In terms of space form geometry, suppose that $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ and $g : M^n \rightarrow \mathbb{Q}_c^{n+p}$ are two isometric immersions. We say that f and g are *Lie equivalent* if their Legendre lifts are Lie equivalent. It is easy to see that, for λ and γ as above, the point $[K]$ on the line $\lambda(x)$ is a curvature sphere of λ at x if and only if the point $\gamma[K]$ is a curvature sphere of the Legendre submanifold $\gamma\lambda$ at x . Furthermore, the principal spaces corresponding to $[K]$ and $\gamma[K]$ are identical. By Theorem 6, two important special cases are when the Lie sphere transformation γ is a Möbius or a parallel transformation. In the first case, suppose that $\gamma = \Phi$ is induced by a conformal diffeomorphism $\phi : \mathbb{Q}_c^{n+p} \rightarrow \mathbb{Q}_c^{n+p}$, where

$$\bar{\mathbb{Q}}_c^{n+p} = \begin{cases} \mathbb{S}_c^{n+p} - \{\sigma\}, & \text{if } c > 0, \\ \mathbb{Q}_c^{n+p}, & \text{if } c \leq 0. \end{cases}$$

If $f : M^n \rightarrow \bar{\mathbb{Q}}_c^{n+p}$ is an isometric immersion and $\tilde{f} = \phi \circ f : M^n \rightarrow \bar{\mathbb{Q}}_c^{n+p}$, then the Legendre lifts $\lambda : UN_f^{n+p-1} \rightarrow \Lambda^{2(n+p)-1}$ of f and $\tilde{\lambda} : UN_{\tilde{f}}^{n+p-1} \rightarrow \Lambda^{2(n+p)-1}$ of \tilde{f} are *Möbius conjugated* in the sense that

$$\tilde{\lambda} = \Phi\lambda\bar{\phi}^{-1},$$

where $\bar{\phi} : UN_f^{n+p-1} \rightarrow UN_{\tilde{f}}^{n+p-1}$ is the unit normal bundle isometry induced by ϕ . In the second case, suppose for simplicity that $\gamma = P_t$ is the spherical parallel transformation given in equation (2.5). Recall that P_t has the effect of adding t to the signed radius of each sphere in \mathbb{S}^d while keeping the center fixed. Suppose that $\lambda : N^{d-1} \rightarrow \Lambda^{2d-1}$ is a Legendre submanifold parametrized by the point sphere and great sphere maps $\{Y_1, Y_{d+3}\}$, as in equation (2.9). Then $P_t\lambda = [W_1, W_{d+3}]$, where

$$W_1 = P_t Y_1 = (\cos t, f, \sin t), \quad W_{d+3} = P_t Y_{d+3} = (-\sin t, \xi, \cos t).$$

Note that W_1 and W_{d+3} are not the point sphere and great sphere maps for $P_t\lambda$. Solving for the point sphere map Z_1 and the great sphere map Z_{d+3} of $P_t\lambda$, we find

$$\begin{aligned} Z_1 &= \cos t W_1 - \sin t W_{d+3} = (1, \cos t f - \sin t \xi, 0), \\ Z_{d+3} &= \sin t W_1 + \cos t W_{d+3} = (0, \sin t f + \cos t \xi, 1). \end{aligned}$$

From this, we see that $P_t\lambda$ has spherical projection and spherical unit normal field given, respectively, by

$$(2.12) \quad \begin{aligned} f_{-t} &= \cos t f - \sin t \xi = \cos(-t) f + \sin(-t) \xi, \\ \xi_{-t} &= \sin t f + \cos t \xi = -\sin(-t) f + \cos(-t) \xi. \end{aligned}$$

The minus sign occurs because P_t takes a sphere with center $f_{-t}(x)$ and radius $-t$ to the point sphere $f_{-t}(x)$. We call $P_t\lambda$ a *parallel submanifold* of λ . Formula (2.12) shows the close correspondence between these parallel submanifolds and the parallel hypersurfaces to or the tubes f_t around f , in the case where $f : M^n \rightarrow \mathbb{S}^{n+p}$ is an isometric immersion with $p = 1$ or $p \geq 2$, respectively. The spherical projection f_t has singularities at the focal points of f , but the parallel submanifold $P_t\lambda$ is still a smooth submanifold of Λ^{2d-1} .

An important theorem, due to Pinkall [21], shows that the number of these singularities is bounded for each $x \in M^{d-1}$, which allows us to obtain several consequences by passing to a parallel submanifold, if necessary, and then applying well-known results concerning immersed hypersurfaces in \mathbb{S}^d . Among them are the important facts that if the dimension m of the principal space T of a given curvature sphere K is constant on an open subset U of M^{d-1} , then the principal distribution T is integrable on U , and if $m > 1$, then the curvature sphere map K is constant along the leaves of the principal foliation T .

A connected submanifold S of M^{d-1} is called a *curvature submanifold* if at each $x \in S$, the tangent space $T_x S$ is equal to some principal space T . For example, if $\dim T$ is constant on an open subset U of M^{d-1} , then each leaf of the principal foliation T is a curvature submanifold on U . Curvature submanifolds are plentiful, since the results of Reckziegel [19] and Singley [43] imply that there is an open dense subset Ω of M^{d-1} on which the multiplicities of the curvature spheres are locally constant. On Ω , each leaf of each principal foliation is a curvature submanifold. On the other hand, it is also possible to have a curvature submanifold S which is not a leaf of a principal foliation, because the multiplicity of the corresponding curvature sphere is not constant on a neighborhood of S . The fact that curvature spheres of multiplicity $m \geq 2$ are constant along the leaves of their corresponding principal foliations can be generalized to the fact that curvature spheres are always constant along corresponding curvature submanifolds of dimension $m \geq 2$. The proof is obtained by invoking a theorem of Ryan [5].

The notion of Dupin submanifold is generalized to the context of Lie sphere geometry by calling a Legendre submanifold *Dupin* if along each curvature submanifold, the corresponding curvature sphere is constant [39]. A Dupin submanifold,

$$\lambda : M^{d-1} \rightarrow \Lambda^{2d-1},$$

is said to be *proper Dupin* if the number k of distinct curvature spheres is constant on M^{d-1} . In this case, we also refer to λ as a k -Dupin submanifold. Of course, Legendre lifts of Dupin hypersurfaces in \mathbb{S}^{n+1} are Dupin in the sense defined here. Indeed, we have the following basic link between the two definitions.

Theorem 7 (Cecil [39]). *Let $f : M^n \rightarrow \mathbb{S}^{n+1}$ be a hypersurface and $\lambda : M^n \rightarrow \Lambda^{2n+1}$ its Legendre lift. Let Y_1 and Y_{n+4} be the point sphere and great sphere maps of λ , respectively. Then the curvature spheres of λ at a point $x \in M^n$ are*

$$[K_i] = [\kappa_i Y_1 + Y_{n+4}], \quad 1 \leq i \leq k,$$

where $\kappa_1, \dots, \kappa_k$ are the distinct principal curvatures at x of f . The principal space of λ corresponding to the curvature sphere $[K_i]$ equals the principal space of f corresponding to the principal curvature κ_i . In particular, the multiplicity of $[K_i]$ equals the multiplicity of κ_i . Moreover, $[K_i]$ is Dupin if and only if κ_i is Dupin.

Proof. This follows immediately from the fact that the point

$$[K] = [\kappa Y_1 + Y_{n+4}]$$

is a curvature sphere of λ at x with corresponding principal vector $X \in T_x M$ if and only if

$$\begin{aligned} \kappa Y_{1*} X + Y_{n+4*} X &= \kappa (0, f_* X, 0) + (0, -f_* AX, 0) \\ &= (0, -f_* (AX - \kappa X), 0) = 0, \end{aligned}$$

in which case

$$K_* X = \kappa_* X Y_1 + \kappa Y_{1*} X + Y_{n+4*} X = \kappa_* X Y_1$$

is collinear with K if and only if $\kappa_* X = 0$, as desired. \square

Pinkall's [20] definition of Dupin submanifolds in higher codimension can also be expressed in terms of their Legendre lifts.

Theorem 8 (Cecil [39]). *Let $f : M^n \rightarrow \mathbb{S}^{n+p}$, $p \geq 2$, be an isometric immersion and $\lambda : UN_f^{n+p-1} \rightarrow \Lambda^{2(n+p)-1}$ its Legendre lift. Let Y_1 and Y_{n+p+3} be the point sphere and great sphere maps of λ , respectively. Then the curvature spheres of λ at a point $(x, \xi) \in UN_f^{n+p-1}$ are*

$$(2.13) \quad [K_i] = [\kappa_i Y_1 + Y_{n+p+3}], \quad 1 \leq i \leq k+1,$$

where $\kappa_1, \dots, \kappa_k$ are the distinct principal curvatures at (x, ξ) of f and $\kappa_{k+1} = \infty$. For $1 \leq i \leq k$, the principal space of λ corresponding to the curvature sphere $[K_i]$ is given by

$$T_i = \{(X, -f_* A_\xi X) : X \in E_i\},$$

where E_i is the principal space of f corresponding to the principal curvature κ_i , while the principal space corresponding to $[K_{k+1}]$ is

$$(2.14) \quad T_{k+1}(x, \xi) = \{(0, \eta) : \eta \in N_f M(x), \eta \cdot \xi = 0\}.$$

In particular, the multiplicity of $[K_i]$ equals the multiplicity of κ_i for $1 \leq i \leq k$, while the multiplicity of $[K_{k+1}]$ is $p-1$. Moreover, for $1 \leq i \leq k$, $[K_i]$ is Dupin if and only if κ_i is Dupin, while $[K_{k+1}]$ is always Dupin.

Proof. Note that

$$[K_{k+1}] = [\infty Y_1 + Y_{n+p+3}] = [Y_1]$$

is simply the point sphere map of λ , which is clearly a curvature sphere with corresponding principal space T_{k+1} given by (2.14) and multiplicity $p-1$, since $p \geq 2$. Moreover, $[K_{k+1}]$ is obviously constant along T_{k+1} and is thus Dupin. On the other hand, as in the case of hypersurfaces we have that the point

$$[K] = [\kappa Y_1 + Y_{n+p+1}], \quad \kappa \neq \infty,$$

is a curvature sphere of λ at (x, ξ) with corresponding principal vector $(X, -f_*A_\xi X + \eta) \in T_{(x, \xi)}UN_f$ if and only if

$$\begin{aligned} (\kappa Y_{1*} + Y_{n+p+1*})(X, -f_*A_\xi X + \eta) &= \kappa(0, f_*X, 0) + (0, -f_*A_\xi X + \eta, 0) \\ &= (0, -f_*(AX - \kappa X) + \eta, 0) = 0, \end{aligned}$$

which implies that $\eta = 0$ and that κ is a principal curvature of f with corresponding principal vector X , as we wished. Furthermore, we now have that

$$\begin{aligned} K_*(X, -f_*A_\xi X) &= \kappa_*(X, -f_*A_\xi X)Y_1 + (\kappa Y_{1*} + Y_{n+p+1*})(X, -f_*A_\xi X) \\ &= \kappa_*(X, -f_*A_\xi X)Y_1, \end{aligned}$$

which again is collinear with K if and only if $\kappa_*(X, -f_*A_\xi X) = 0$. \square

As an immediate consequence of the above two results, we get a bijective correspondence between k -Dupin submanifolds of codimension $p \geq 2$ (respectively, k -Dupin hypersurfaces) in space forms and $(k+1)$ -Dupin (respectively, k -Dupin) Legendre lifts.

Corollary 1. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ be an isometric immersion and λ its Legendre lift. If $p \geq 2$ (respectively, $p = 1$), then f is k -umbilical if and only if λ is $(k+1)$ -umbilical (respectively, k -umbilical). Furthermore, f is k -Dupin if and only if λ is $(k+1)$ -Dupin (respectively, k -Dupin).*

Another interesting application of Theorems 7 and 8 is the invariance of k -Dupin submanifolds under conformal transformations of and between ambient space forms.

Corollary 2. *Let $f : M^n \rightarrow \bar{\mathbb{Q}}_c^{n+p}$ be a k -umbilical isometric immersion and let $\phi : \bar{\mathbb{Q}}_c^{n+p} \rightarrow \bar{\mathbb{Q}}_c^{n+p}$ be a conformal diffeomorphism. Then, the composition $\tilde{f} = \phi \circ f : M^n \rightarrow \bar{\mathbb{Q}}_c^{n+p}$ is also k -umbilical. Moreover, if f is k -Dupin, so is \tilde{f} .*

Proof. Since the Legendre lifts of f and \tilde{f} are Möbius conjugated by ϕ , this follows immediately from the preceding corollary together with the fact that Lie sphere transformations map curvature spheres to curvature spheres and also preserve the Dupin condition. \square

We close this subsection with a quick review of the theory Dupin submanifolds in the context of Lie sphere geometry. Pinkall [21] introduced three constructions for obtaining a Dupin hypersurface in \mathbb{R}^{n+m} from a Dupin hypersurface in \mathbb{R}^{n+1} , namely, tubes, cylinders and submanifolds of revolution, which give birth to a $(k+1)$ -Dupin hypersurface out of a lower-dimensional k -Dupin hypersurface. These standard constructions can also be formulated in the context of Legendre submanifolds and apply even to Legendre lifts of submanifolds of higher codimension, although in this case the number of distinct curvature spheres remains the same.

Remark 13. *When Pinkall introduced his constructions, he also listed cones, which turn out to be locally Lie equivalent to tubes.*

A Dupin submanifold obtained from a lower-dimensional Dupin submanifold via one of Pinkall's standard constructions is said to be *reducible*. More generally, a Dupin submanifold which is locally Lie equivalent to such a Dupin submanifold is called *reducible*. Pinkall [21] formulated the following simple Lie sphere geometric criterion for reducibility.

Theorem 9 (Pinkall [21]). *A connected proper Dupin submanifold $\lambda : N^{d-1} \rightarrow \Lambda^{2d-1}$ is reducible if and only if there exists a curvature sphere $[K]$ of λ that lies in a linear subspace of \mathbb{P}^{d+2} of codimension at least two.*

Remark 14. *In [22], Dajczer, Florit and Tojeiro studied reducibility in the context of Riemannian geometry. They formulated a concept of weak reducibility for proper Dupin submanifolds that have a flat normal bundle, including proper Dupin hypersurfaces. For hypersurfaces, their definition can be formulated as follows. A proper Dupin hypersurface $f : M^n \rightarrow \mathbb{Q}_c^{n+1}$ is said to be weakly reducible if, for some principal curvature κ_i with corresponding principal space E_i , the orthogonal complement E_i^\perp is integrable. Dajczer, Florit and Tojeiro show that if a proper Dupin hypersurface $f : M^n \rightarrow \mathbb{Q}_c^{n+1}$ is Lie equivalent to a $(k+1)$ -Dupin hypersurface that is obtained via one of the standard constructions from a k -Dupin hypersurface, then f is weakly reducible. Thus, reducible implies weakly reducible for such hypersurfaces.*

However, one can show that the open subset U of a tube over the Veronese surface regarded in \mathbb{S}^5 (rather than in \mathbb{S}^4) on which there are three principal curvatures at each point is reducible but not weakly reducible, because none of the orthogonal complements of the principal spaces is integrable. Of course, U is not constructed from a 2-Dupin submanifold, but rather one 3-Dupin.

A proper Dupin hypersurface with two distinct curvature spheres of respective multiplicities p and q is called a *cyclide of Dupin of characteristic (p, q)* . These are the simplest Dupin submanifolds after the spheres, and they were first studied in \mathbb{R}^3 by Dupin [44] in 1822. An example of a cyclide of Dupin of characteristic $(1, 1)$ in \mathbb{R}^3 is a torus of revolution. The cyclides were studied by many prominent mathematicians in the nineteenth century, including Liouville [45], Cayley [46], and Maxwell [47], whose paper contains stereoscopic figures of the various types of cyclides. The long history of the classical cyclides of Dupin is given in Lilienthal [48]. (See also Banchoff [49], Cecil [50], Klein [51], Darboux [52]–[53], Blaschke [54], Eisenhart [55], Hilbert and Cohn-Vossen [56], Fladt and Baur [57], and Cecil and Ryan [58], [59], for more on the classical cyclides.) For a consideration of the cyclides in the context of computer graphics, see Degen [60], Pratt [61]–[62], Srinivas and Dutta [63]–[64]–[65]–[66], and Schrott and Odehnal [67]. For cyclides of Dupin in \mathbb{R}^3 , it was known in the nineteenth century that every connected Dupin cyclide is Möbius equivalent to an open subset of a surface of revolution obtained by revolving a profile circle $\mathbb{S}^1 \subset \mathbb{R}^2$ about an axis $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$. The profile circle is allowed to intersect the axis, thereby introducing Euclidean singularities. However, the corresponding Legendre map into the space of contact elements in \mathbb{R}^3 is an immersion. Higher-dimensional cyclides of Dupin appeared in the study of isoparametric hypersurfaces in spheres. Cartan knew that an isoparametric hypersurface in a sphere with two curvature spheres must be a standard product of spheres,

$$\mathbb{S}_{c_1}^d \times \mathbb{S}_{c_2}^{n-d} \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{d+1} \times \mathbb{R}^{n-d+1} = \mathbb{R}^{n+2}, \quad c_1^{-1} + c_2^{-1} = 1.$$

Cecil and Ryan [68] showed that a compact cyclide of Dupin M^n embedded in \mathbb{S}^{n+1} must be Möbius equivalent to a standard product of spheres. The proof, however, uses the compactness of M^n in an essential way. Later, Pinkall [21] used Lie sphere geometric techniques to show that any two cyclides of Dupin of the same characteristic are locally Lie equivalent, which leads to a local classification of the higher-dimensional cyclides of Dupin that is analogous to the classical result,

namely, every cyclide of Dupin is locally conformally congruent to a rotational hypersurface with a sphere as profile.

Remark 15. *The latter result also follows easily from the Moore-type decomposition theorem for conformal immersions of a warped product of Riemannian manifolds due to Tojeiro [69].*

The case of 3-Dupin hypersurfaces was studied in the paper of Cecil and Jensen [32]. Their result can be stated as follows, where the case $d = 4$ is credited to Pinkall [31].

Theorem 10 (Pinkall [31], Cecil-Jensen [32]). *Let $\lambda : N^{d-1} \rightarrow \Lambda^{2d-1}$, $d \geq 4$, be a connected irreducible 3-Dupin submanifold. Then $d = 3 \cdot 2^{\mu-1} + 1$, where $\mu = 1, 2, 3, 4$, each principal curvature has the same multiplicity $2^{\mu-1}$ and λ is Lie equivalent to the Legendre lift of a 3-isoparametric hypersurface in $\mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$.*

Remark 16. *In [23], Cartan showed that a 3-isoparametric hypersurface in the sphere is always a tube $\psi_{\mathbb{F}_t} : UN_{\psi_{\mathbb{F}}}^{3 \cdot 2^{\mu-1}} \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ over a standard embedding $\psi_{\mathbb{F}} : \mathbb{FP}^2 \rightarrow \mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ of one of the projective planes \mathbb{FP}^2 , for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\mu = 1, 2, 3, 4$, respectively. Since $\psi_{\mathbb{F}_t}$ is exactly the spherical projection of the parallel submanifold $P_{-t}\lambda$ of the Legendre lift $\lambda : UN_{\psi_{\mathbb{F}}}^{3 \cdot 2^{\mu-1}} \rightarrow \Lambda^{3 \cdot 2^{\mu} + 1}$ of $\psi_{\mathbb{F}}$, we have that $P_{-t}\lambda$ is the actual Legendre lift of $\psi_{\mathbb{F}_t}$. In other words, Legendre lifts of 3-isoparametric hypersurfaces and those of their corresponding standard embeddings of the projective planes are parallel, and in particular Lie equivalent.*

2.3. Generalized cylinders and envelopes of hyperspheres. This subsection is devoted to discussing the concepts of generalized cylinders and envelopes of hyperspheres.

Let $g : L^k \rightarrow \mathbb{Q}_c^{n+p}$ be an isometric immersion with a parallel flat normal subbundle \mathcal{V} of rank $n - k$. The *generalized $(n - k)$ -cylinder* over g determined by \mathcal{V} is the n -dimensional submanifold parametrized by means of the exponential map of \mathbb{Q}_c^{n+p} as

$$\gamma \in \mathcal{V} \mapsto \exp_{g(\pi(\gamma))}^c(\gamma),$$

where $\pi : \mathcal{V} \rightarrow L^k$ is the projection. It was shown by Dajczer-Florit-Tojeiro [22] that generalized cylinders are the only submanifolds that carry a relative nullity distribution with integrable conullity.

Let $f : M^n \rightarrow \mathbb{Q}_c^{n+1}$ be a connected orientable hypersurface, we denote by η the unit normal to f which gives the orientation of M^n . Following Asperti-Dajczer [22], f is said to be a *k -parameter envelope of hyperspheres* (briefly, *k -PES*) if it carries a nonzero Dupin principal curvature κ of multiplicity $n - k$. Actually, in [22] the authors restrict themselves to the case $k \geq 2$, in which the Dupin condition is automatically satisfied. Classically, a Euclidean k -PES is locally a solution

$$f = f(u_1, \dots, u_k, t_1, \dots, t_{n-k})$$

in \mathbb{R}^{n+1} of the system below:

$$(2.15) \quad \begin{aligned} (a) \quad & \|f - g\|^2 = r^2, \\ (b) \quad & \left\langle f - g, \frac{\partial g}{\partial u_i} \right\rangle = -r \frac{\partial r}{\partial u_i}, \quad i = 1, \dots, k, \end{aligned}$$

where $g : L^k \rightarrow \mathbb{R}^{n+1}$ is an isometric immersion, $g = g(u_1, \dots, u_k)$, and $r \in C^\infty(L)$ is a non-vanishing function. Geometrically it means that f is the envelope of the k -parameter family of hyperspheres given by (a): the limit of the intersection of neighboring hyperspheres that approach each other are $(n-k)$ -spheres that generate the envelope.

Let g and r be as above and let t_1, \dots, t_{n-k} be parameters of the unit $(n-k)$ -sphere centered at the origin of a Euclidean $(n-k+1)$ -space. Set

$$(2.16) \quad f(u_1, \dots, u_k, t_1, \dots, t_{n-k}) = g - r \nabla r - r \sqrt{1 - |\nabla r|^2} \varphi(t_1, \dots, t_{n-k}),$$

where ∇r is the gradient of r and the vector φ has origin at the point $\gamma = g - r \nabla r$ and describes a unit sphere in the affine $(n-k+1)$ -plane through γ orthogonal to g . It was shown by Asperti-Dajczer [34] that the hypersurface given by (2.16) satisfies the system (2.15) and is (away from singular points) a k -PES. Conversely, every Euclidean k -PES satisfies system (2.15) and is locally of the form (2.16), for $r = 1/\kappa$. Although their proof is carried out in Euclidean space, it can be easily adapted in nonflat space forms in order to get a similar parametrization of k -PES also in these ambient spaces.

Envelopes of hyperspheres have the following natural generalization in higher codimension. For simplicity, we discuss just the Euclidean case, the others being analogous. Given smooth maps $g : M^n \rightarrow \mathbb{R}^{n+p}$ and $r \in C^\infty(M)$, the family of hyperspheres

$$y \in M^n \mapsto S(g(y); r(y))$$

centered at $g(y) \in \mathbb{R}^{n+p}$ with radius $r(y)$ is said to be a *congruence* of hyperspheres. It is said to be a *k-parameter congruence* of hyperspheres if g has rank k everywhere and $\ker g_*(y) \subset \ker r_*(y)$ for all $y \in M^n$. An immersion $f : M^n \rightarrow \mathbb{R}^{n+p}$ is said to *envelope* the congruence of hyperspheres along M^n determined by $g : M^n \rightarrow \mathbb{R}^{n+p}$ and $r \in C^\infty(M)$ if

$$f_* T_y M \subset T_{f(y)} S(g(y); r(y))$$

for all $y \in M^n$.

A principal normal vector field η is said to be *Dupin* if η is parallel in the normal connection along E_η . One can prove that if $f : M^n \rightarrow \mathbb{R}^{n+p}$ is an isometric immersion that carries a Dupin principal normal vector field η of multiplicity $n-k$ then its focal submanifold

$$(2.17) \quad g = f + \frac{1}{\|\eta\|^2} \eta$$

and its curvature radius $r = \|\eta\|^{-1}$ determine an $(n-k)$ -parameter congruence of hyperspheres that is enveloped by f . Conversely, every isometric immersion $f : M^n \rightarrow \mathbb{R}^{n+p}$ that envelopes an $(n-k)$ -parameter congruence of hyperspheres along M^n determined by $g : M^n \rightarrow \mathbb{R}^{n+p}$ and $r \in C^\infty(M)$ carries a Dupin principal normal vector field

$$\eta = \frac{1}{r^2} (g - f)$$

of multiplicity k .

Remark 17. Let $f : M^n \rightarrow \mathbb{R}^{n+1}$ be a k -PES. We say that f is a special k -parameter envelope of hyperspheres (briefly, *k-SPES*) if its Dupin curvature of

multiplicity $n - k$ has integrable conullity. In [34, Theorem (1.7)], Asperti and Dajczer stated the following result on k -SPES:

Let $f : M^n \rightarrow \mathbb{R}^{n+1}$ be a k -PES. Then f is a k -SPES if and only if $g : L^k \rightarrow \mathbb{R}^{n+1}$ has flat normal bundle and $\nabla r(x) \in \Delta(x)$ for all $x \in L^k$, where $r = 1/\kappa$.

However, we point out that this result is not true. The counterexamples are the compact embedded cyclides of Dupin $f : M^n \rightarrow \mathbb{R}^{n+1}$ of characteristic (p, q) . In this case f is both a p -SPES and a q -SPES. Since M^n is compact then also L^k is compact and thus the relative nullity Δ of the focal submanifold g is trivial because g is real analytic (this is so because f is real analytic due to [68]). If the above statement were true the second condition would imply that these cyclides of Dupin are always isoparametric.

The mistake happens in the first step of their proof. The authors consider the general parametrization for f given in equation (2.16). If in addition f is a k -SPES, the authors claim that the local coordinates

$$(2.18) \quad (u_1, \dots, u_k, t_1, \dots, t_{n-k})$$

in parametrization (2.16) can be chosen such that (u_1, \dots, u_k) are coordinates for the leaves of the Dupin conullity E_κ^\perp . Since E_κ and E_κ^\perp are integrable, there do exist local coordinates (2.18) such that (t_1, \dots, t_{n-k}) and (u_1, \dots, u_k) are coordinates for the leaves of E_κ and E_κ^\perp , respectively, but we cannot pick them without breaking the independence of φ from the variables u_1, \dots, u_k . In other words, if the local coordinates (2.18) are chosen as above, then parametrization (2.16) becomes

$$f(u_1, \dots, u_k, t_1, \dots, t_{n-k}) = g - r\nabla r - r\sqrt{1 - |\nabla r|^2}\varphi(u_1, \dots, u_k, t_1, \dots, t_{n-k}).$$

3. PROOFS

In this section we prove our four main results stated in the introduction of this article. We start with the proof of Theorem 2, which is a simple application of Cartan's classification of 3-isoparametric hypersurfaces and the following result.

Theorem 11 (Heintze-Olmos-Thorbergsson [18]). *A submanifold of a space form has constant principal curvatures if and only if it is either isoparametric or a focal manifold to an isoparametric submanifold.*

For a discussion of focal manifolds of an isoparametric submanifold, its related Coxeter group, etc., see [2].

Proof of Theorem 2. Since f has constant principal curvatures, as a consequence of Theorem 11, f is either isoparametric or a focal manifold of an isoparametric submanifold. If f is isoparametric, then it has flat normal bundle, which contradicts the assumptions that f is 2-CPC and $p \geq 2$. Thus, f is a focal submanifold of an isoparametric submanifold $g : N^{n+q} \rightarrow \mathbb{Q}_c^{n+p}$. Then from the proof of Theorem 11 it follows that g is a holonomy tube f_{ξ_x} through a principal vector $\xi_x \in N_f M(x)$. We claim that g is an isoparametric hypersurface of a geodesic hypersphere. In fact, from [2] it is enough to show that the codimension of a principal orbit $\mathcal{H}ol_x(\nabla^\perp) \cdot \xi_x$ is one. Indeed, if it is greater than one, we will obtain a contradiction. Note first that the Ricci equation implies $[A_\xi, A_\eta] = 0$ for any two normal vectors ξ, η to a principal orbit of the action of the normal holonomy group. So, we can perform a simultaneous diagonalization of the family $\{A_\xi : \xi \in \text{the normal space to a principal orbit}\}$. Then, given any two linearly

independent normal vectors to a principal orbit, it is possible to find a nonzero umbilical linear combination of them, contradicting the assumption that f is 2-CPC.

Thus, g is an isoparametric hypersurface of a geodesic hypersphere and the isometric immersion f is a focal submanifold of g , i.e., $f = g_{\xi/\kappa}$, where ξ is the unit normal of g and κ a constant principal curvature of g . Since the shape operator A_{ξ} of f has two eigenspaces for any normal vector ξ then the “Tube formula” (see [2]) implies that the shape operator of g must have three principal curvatures.

So from a theorem of Elie Cartan [23] it follows that f must be one of the cited embeddings (for a beautiful proof of this Cartan’s theorem see [27]).

Conversely, it is well known that the standard embeddings of the projective planes are 2-CPC. \square

We now go on to prove our main Theorem 3, for which we need three more lemmas. The first one asserts that the Legendre lift of a 2-Dupin submanifold of codimension greater than one is irreducible in the sense of Lie sphere geometry.

Lemma 3. *Let $f : M^{2m} \rightarrow \mathbb{S}^{2m+p}$, $p \geq 2$, be a connected 2-Dupin isometric immersion. Then, its Legendre lift $\lambda : UN_f^{2m+p-1} \rightarrow \Lambda^{2(2m+p)-1}$ is an irreducible 3-Dupin submanifold with two principal curvatures of multiplicity m and the other $p-1$.*

Proof. The fact that λ is 3-Dupin follows directly from Theorem 8. Moreover, the infinite principal curvature κ_3 has multiplicity $p-1$ and the two others have the same multiplicity m by Lemma 2.

It remains to be shown that λ is irreducible. Let $[K_i]$, $1 \leq i \leq 3$, be the curvature spheres of λ given by (2.13), with $n = 2m$. By Theorem 9 all there is left to prove is that no $[K_i]$ lies in a linear subspace of \mathbb{P}^{2m+p+2} of codimension higher than one. For the point sphere map $[K_3]$, this follows easily from the fact that f is conformally substantial. In fact, if $[K_3]$ lied in a \mathbb{P}^{2m+p} , then there would be a nonzero vector $v = (v_0, w, 0) \in \mathbb{R}_2^{2m+p+3}$ other than $(0, 0, 1)$ that would be orthogonal to $\text{span}\{K_3\}$, so that

$$f(x) \cdot w = v_0,$$

which is the equation of a hyperplane in \mathbb{R}^{2m+p+1} . Thus, $f(M)$ would be contained in a hypersphere \mathbb{S}_c^{2m+p-1} , contradicting the fact that f is conformally substantial.

To rule out the other two cases, note that, by Lemma 2, $\kappa_i(x, -\xi) = -\kappa_j(x, \xi)$, $1 \leq i \neq j \leq 2$, and thus

$$\begin{aligned} [K_i(x, \xi)] &= [\kappa_i Y_1(x, \xi) + Y_{2m+p+3}(x, \xi)] = [(\kappa_i, \kappa_i f(x) + \xi, 1)], \\ [K_i(x, -\xi)] &= [-\kappa_j Y_1(x, -\xi) + Y_{2m+p+3}(x, -\xi)] = [(-\kappa_j, -\kappa_j f(x) - \xi, 1)], \end{aligned}$$

with $\kappa_i = \kappa_i(x, \xi)$ and similarly for κ_j . Adding these two equations, we get that

$$\begin{aligned} [K_i(x, \xi) + K_i(x, -\xi)] &= [(\kappa_i - \kappa_j, (\kappa_i - \kappa_j) f(x), 2)] \\ &= \left[\left(\kappa_1 - \kappa_2, (\kappa_1 - \kappa_2) f(x), (-1)^{i+1} 2 \right) \right] \in \text{span}\{K_{t_i}\}, \end{aligned}$$

for all $x \in M^{2m}$, $1 \leq i \leq 2$. Then the same argument used to discard $[K_3]$ gives that f would reduce conformal codimension again if some $[K_i]$ lied in a linear subspace \mathbb{P}^{2m+p} , and this concludes the proof. \square

Remark 18. *In the proof of Theorem 5 we will need a slightly more general version of Lemma 3. Suppose that we are given a connected umbilic-free weakly 2-umbilical*

isometric immersion $f : M^{2m} \rightarrow \mathbb{S}^{2m+p}$, $p \geq 2$. Then, restricting the Legendre lift λ of f to the “umbilic free” part UN_f^0 of UN_f (i.e., the set of $(x, \xi) \in UN_f$ such that A_ξ is not a multiple of I), we still have by Theorem 8 and Remark 10-(ii) that $\lambda|_{UN_f^0}$ is 3-Dupin with the infinite principal curvature κ_3 of multiplicity $p-1$ and the two others of the same multiplicity m . Moreover, if we have in addition that f is conformally substantial, then an entirely analogous proof shows that $\lambda|_{UN_f^0}$ is also irreducible.

The next two lemmas are related. Lemma 4 somehow characterizes the standard embeddings of the four projective planes as the only 2-umbilical submanifolds with codimension greater than one in space forms which can be focalized from a 3-Dupin hypersurface in two different ways. Lemma 5 is the Lie sphere geometric counterpart of Lemma 4 in terms of Legendre lifts.

Lemma 4. *Let $f : M^{2m} \rightarrow \mathbb{Q}_c^{2m+p}$, $p \geq 2$, be a 2-umbilical isometric immersion. If the tube $f_t : UN_f^{2m+p-1} \rightarrow \mathbb{Q}_c^{2m+p}$ is nowhere an immersion for some $t \neq k\pi$, $k \in \mathbb{Z}$, if $c > 0$ or $t \neq 0$ if $c \leq 0$, then actually $c > 0$,*

$$m = p - 1 = 2^{\mu-1}, \quad \mu = 1, 2, 3, 4,$$

and, up to congruence and the similarity θ_c , we have that

$$f(M) \subset \psi_{\mathbb{F}}(\mathbb{FP}^2),$$

where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, according to whether $\mu = 1, 2, 3, 4$, respectively.

Proof. We will basically show that the assumption implies that f is 2-unipotent, from which the statement will then follow, by Theorem 1.

Since the computations in all three cases are similar, we restrict attention to the spherical one ($c = 1$). Then f_t is given by

$$(3.1) \quad f_t = \cos t f + \sin t \xi.$$

The singularities of f_t take place exactly at the focal points of f . Indeed, differentiating (3.1), we obtain

$$(3.2) \quad f_{t*}(X, -f_*A_\xi X + \eta) = f_*(\cos t X - \sin t A_\xi X) + \sin t \eta,$$

for all $X \in T_x M$ and all $\eta \in N_f M(x)$ orthogonal to ξ . Choosing $X = 0$ in (3.2), we get

$$(3.3) \quad f_{t*}(0, \eta) = \sin t \eta,$$

but since $t \neq k\pi$, $k \in \mathbb{Z}$, it follows that $\sin t \neq 0$, and thus we conclude that every $\eta \perp \xi$ lies in the range of f_{t*} at (x, ξ) . On the other hand, taking now $\eta = 0$ in (3.2), we have

$$(3.4) \quad f_{t*}(X, -f_*A_\xi X) = f_*(\cos t X - \sin t A_\xi X),$$

and so the assumption that f_t is nowhere an immersion, together with the previous conclusion, forces (3.4) to vanish for some $X(x, \xi)$ at every $(x, \xi) \in UN_f^{2m+p-1}$. In particular, it follows that $\cot t$ is a common principal curvature of all A_ξ , $(x, \xi) \in UN_f^{2m+p-1}$. Hence, we have that f is 2-unipotent by Lemma 1. The result then follows immediately from Theorem 1. \square

Lemma 5. *Let $f : M^{2m} \rightarrow \mathbb{Q}_c^{2m+p}$, $p \geq 2$, be a 2-umbilical isometric immersion and let $\lambda : UN_f^{2m+p-1} \rightarrow \Lambda^{2(2m+p)-1}$ be its Legendre lift. If the spherical projection of $P_t \lambda$ is nowhere an immersion for some spherical, Euclidean or hyperbolic parallel transformation $P_t \neq \pm I$, then*

$$m = p - 1 = 2^{\mu-1}, \quad \mu = 1, 2, 3, 4,$$

and, up to congruence and the stereographic projection π_c if $c \leq 0$ or similarity θ_c if $c > 0$, we have that

$$f(M) \subset \psi_{\mathbb{F}}(\mathbb{FP}^2),$$

where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, according to whether $\mu = 1, 2, 3, 4$, respectively.

Proof. After possibly composing f with the stereographic projection, we are free to work in the ambient space to which the action of P_t is best suited. More precisely, we can assume that $c = 1, 0, -1$ depending on whether P_t is spherical, Euclidean or hyperbolic, respectively. Then the spherical projection of $P_t \lambda$ is simply the tube f_{-t} , and the result follows from the previous lemma. \square

3.1. Proof of Theorem 3. To prove the theorem all one has to do is put together the previous lemmata.

Proof. By Lemma 2, we have that $n = 2m$. It follows from Lemma 3 that the Legendre lift $\lambda : UN_f^{2m+p-1} \rightarrow \Lambda^{2(2m+p)-1}$ of f is a connected irreducible 3-Dupin submanifold with two principal curvatures of multiplicity m and the other $p - 1$. Thus, we are under the assumptions of Theorem 10 and therefore we conclude that $p - 1 = m = 2^{\mu-1}$ for $\mu = 1, 2, 3, 4$ (hence $2m + p = 3 \cdot 2^{\mu-1} + 1 = 4, 7, 13, 25$, respectively) and λ is Lie equivalent to an open subset of the Legendre lift of a 3-isoparametric hypersurface in $\mathbb{S}^4, \mathbb{S}^7, \mathbb{S}^{13}, \mathbb{S}^{25}$, or equivalently, of that of the corresponding standard embedding $\psi_{\mathbb{F}}$ of the projective plane \mathbb{FP}^2 , for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively (see Remark 16). In other words, there exists a Lie sphere transformation γ such that the spherical projection \tilde{f} of

$$(3.5) \quad \tilde{\lambda} = \gamma \lambda$$

is a submersion of rank $n = 2^{\mu}$ onto an open subset of the image of $\psi_{\mathbb{F}}$. On the other hand, we know from Theorem 6 that any such γ can be written as

$$\gamma = \Phi_1 P_t \Phi_2,$$

where Φ_1 and Φ_2 are Möbius transformations and P_t is some spherical, Euclidean or hyperbolic parallel transformation. Hence, taking Φ_1^{-1} on both sides of (3.5) we have

$$\Phi_1^{-1} \tilde{\lambda} = P_t \Phi_2 \lambda.$$

Now let ϕ_i be the conformal transformation of $\mathbb{S}^{3 \cdot 2^{\mu-1} + 1}$ induced by Φ_i , $i = 1, 2$. Since λ is the Legendre lift of f , it follows that

$$\Phi_2 \lambda = \bar{\lambda} \bar{\phi}_2,$$

where $\bar{\phi}_2 : UN_{\phi_2 \circ f}^{3 \cdot 2^{\mu-1}} \rightarrow UN_{\phi_2 \circ f}^{3 \cdot 2^{\mu-1}}$ is the unit normal bundle isometry induced by ϕ_2 and $\bar{\lambda} : UN_{\phi_2 \circ f}^{3 \cdot 2^{\mu-1}} \rightarrow \Lambda^{3 \cdot 2^{\mu} + 1}$ is the Legendre lift of the composition $\phi_2 \circ f$. Moreover, the spherical projection of $\Phi_1^{-1} \tilde{\lambda}$ is simply $\phi_1^{-1} \circ \tilde{f}$, and thus is also a submersion of rank 2^{μ} onto its image. Hence, so must be that of $P_t \bar{\lambda}$ and, in particular, is nowhere an immersion, from which it follows, by Lemma 5, that

already $\phi_2 \circ f(M) \subset \psi_{\mathbb{F}}(\mathbb{FP}^2)$ up to congruence, or else $P_t = \pm I$. In both cases we conclude that

$$(3.6) \quad f(M) \subset \psi_{\mathbb{F}}(\mathbb{FP}^2)$$

after a conformal motion, which completes the proof of the direct statement.

The converse follows immediately from the conformal invariance of the class of 2-Dupin submanifolds given by Corollary 2. \square

3.2. Proofs of Theorems 4 and 5.

Here we prove Theorem 4.

Proof of Theorem 4. If $n = 2$, then $p \leq 2$, otherwise there are umbilical directions, since the vector subspace of real symmetric 2×2 matrices is 3-dimensional. This is case (iv) in the statement. Now we can assume $n \geq 3$. If every principal curvature has multiplicity greater than or equal to two, then f is automatically 2-Dupin, so cases (i) and (ii) follow from Theorem 3 and Pinkall's classification of the cyclides, respectively. Now we can assume that one of the principal curvatures has multiplicity one. From Lemma 10 it follows that $p = 1$. Let κ be the principal curvature of multiplicity $n - 1$. Then, on each connected component of the open subset where $\kappa \neq 0$, we have by definition that f is a 1-PES. Finally, on each connected component of the interior of the closed subset where $\kappa = 0$, we have by the result of Dajczer-Florit-Tojeiro [22] that f is a generalized $(n - 1)$ -cylinder (see Subsection 2.3). \square

Now we prove Theorem 5.

Proof of Theorem 5. In the case $n = 2$ there is nothing to be done. So we assume $n \geq 3$. Assume that the open subset

$$V = \{x \in M^n : R^\perp(x) \neq 0\}$$

is nonempty. By Remark 10-(ii), we have that $n = 2m$, with $m \geq 2$.

Take a connected component U of V , let q be the conformal codimension of $f|_U$ and regard $f|_U$ as an isometric immersion into a totally umbilical submanifold \mathbb{S}_c^{2m+q} of \mathbb{S}^{2m+p} , with $c \geq 1$. Then Remark 18 implies that its umbilic-free Legendre lift $\lambda : UN_f^0 \rightarrow \Lambda$ is a connected irreducible 3-Dupin submanifold with two curvature spheres of multiplicity m and the other $q - 1$. Thus, we are under the assumptions of Theorem 10 and therefore we conclude that $q - 1 = m = 2^{\mu-1}$ for $\mu = 2, 3, 4$ (hence $2m + q = 3 \cdot 2^{\mu-1} + 1 = 7, 13, 25$, respectively) and λ is Lie equivalent (by a Lie sphere transformation γ) to an open subset of the Legendre lift of the standard embedding $\psi_{\mathbb{F}}$ of the projective plane \mathbb{FP}^2 , for $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. As in the proof of Theorem 3, by decomposing γ according to Theorem 6, we get that

$$(3.7) \quad f(U) \subset \text{Im } \psi_{\mathbb{F}}$$

after a conformal motion.

Finally, we claim that $U = M$. Assume otherwise and take $x \in \partial U$. Since $\psi_{\mathbb{F}}$ is homogeneous and has nonflat normal bundle, it follows that $\|R_{\psi_{\mathbb{F}}}^\perp\| = \kappa$ is constant and positive, but then (3.7) yields

$$\|R_f^\perp(x)\| = \kappa > 0,$$

which is a contradiction with $R_f^\perp(x) = 0$. Thus the claim is proved and so

$$(3.8) \quad \text{Im } f \subset \text{Im } \psi_{\mathbb{F}},$$

which gives us case (i) in the statement.

Now we can assume that $R^\perp = 0$ everywhere. If the two principal normals η_1 and η_2 of f have multiplicity ≥ 2 , then they are automatically Dupin, and hence the orthogonal net $\mathcal{E} = (E_1, E_2)$ of their principal distributions is what has been called a CWP-net in [69], which implies that M^n is locally conformal to a warped product. Moreover, since the second fundamental form α of f is adapted to \mathcal{E} (i.e., $\alpha(E_1, E_2) = 0$) and also the leaves of E_i , $1 \leq i \leq 2$, are extrinsic spheres of \mathbb{S}^{n+p} , case (ii) then follows directly from the main theorem in [69].

Finally, case (iii) i.e. when one of the multiplicities is one, is obtained from the considerations of Subsection 2.3 in a similar way as in the proof of Theorem 4. \square

REFERENCES

- [1] M. Dajczer, *Submanifolds and isometric immersions*. Mathematics lecture series. Publish or Perish, Incorporated, 1990. <https://books.google.it/books?id=zK3vAAAAAAAJ>.
- [2] J. Berndt, S. Console, and C. E. Olmos, *Submanifolds and holonomy*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, second ed., 2016. <http://dx.doi.org/10.1201/b19615>.
- [3] T. E. Cecil and P. J. Ryan, “Focal sets of submanifolds,” *Pacific J. Math.* **78** no. 1, (1978) 27–39. <http://projecteuclid.org/euclid.pjm/1102806296>.
- [4] K. Nomizu, “Characteristic roots and vectors of a differentiable family of symmetric matrices,” *Linear and Multilinear Algebra* **1** no. 2, (1973) 159–162, <http://dx.doi.org/10.1080/03081087308817014>. <http://dx.doi.org/10.1080/03081087308817014>.
- [5] P. J. Ryan, “Homogeneity and some curvature conditions for hypersurfaces,” *Tohoku Math. J. (2)* **21** no. 3, (1969) 363–388. <http://dx.doi.org/10.2748/tmj/1178242949>.
- [6] A. J. Di Scala, “On submanifolds whose shape operator is unipotent,” *Journal of Geometry* **85** no. 1, (2006) 32–34. <http://dx.doi.org/10.1007/s00022-006-0039-z>.
- [7] J. Eells, “On equivariant harmonic maps,” in *Proceedings of the 1981 Shanghai symposium on differential geometry and differential equations (Shanghai/Hefei, 1981)*, pp. 55–73. Science Press, Beijing, 1984.
- [8] D. Gromoll and K. Grove, *One-Dimensional Metric Foliations in Constant Curvature Spaces*, pp. 165–168. Springer Berlin Heidelberg, Berlin, Heidelberg, 1985. http://dx.doi.org/10.1007/978-3-642-69828-6_11.
- [9] Q. M. Wang, “Isoparametric maps of Riemannian manifolds and their applications,” in *Advances in science of China. Mathematics, Vol. 2*, Wiley-Intersci. Publ., pp. 79–103. Wiley, New York, 1986. <http://dx.doi.org/10.1093/plankt/8.1.79>.
- [10] C. E. Harle, “Isoparametric families of submanifolds,” *Boletim da Sociedade Brasileira de Matemática - Bulletin/Brazilian Mathematical Society* **13** no. 2, (1982) 35–48. <http://dx.doi.org/10.1007/BF02584674>.
- [11] W. Strübing, “Isoparametric submanifolds,” *Geometriae Dedicata* **20** no. 3, (1986) 367–387. <http://dx.doi.org/10.1007/BF00149586>.
- [12] C.-L. Terng, “Isoparametric submanifolds and their Coxeter groups,” *J. Differential Geom.* **21** no. 1, (1985) 79–107. <http://projecteuclid.org/euclid.jdg/1214439466>.
- [13] G. Thorbergsson, “A survey on isoparametric hypersurfaces and their generalizations,” in *Handbook of differential geometry, Vol. I*, pp. 963–995. North-Holland, Amsterdam, 2000. [http://dx.doi.org/10.1016/S1874-5741\(00\)80013-8](http://dx.doi.org/10.1016/S1874-5741(00)80013-8).
- [14] R. S. Palais and C.-L. Terng, *Critical point theory and submanifold geometry*, vol. 1353 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
- [15] G. Thorbergsson, “Isoparametric Foliations and their Buildings,” *Annals of Mathematics* **133** no. 2, (1991) 429–446. <http://www.jstor.org/stable/2944343>.
- [16] C. Olmos, “Homogeneous submanifolds of higher rank and parallel mean curvature,” *J. Differential Geom.* **39** no. 3, (1994) 605–627. <http://projecteuclid.org/euclid.jdg/1214455074>.

- [17] C. OLMOS and R. RIAÑO-RIAÑO, “Normal holonomy of orbits and Veronese submanifolds,” *J. Math. Soc. Japan* **67** no. 3, (07, 2015) 903–942.
<http://dx.doi.org/10.2969/jmsj/06730903>.
- [18] E. HEINTZE, C. OLMOS, and G. THORBERGSSON, “SUBMANIFOLDS WITH CONSTANT PRINCIPAL CURVATURES AND NORMAL HOLONOMY GROUPS,” *International Journal of Mathematics* **02** no. 02, (1991) 167–175,
<http://www.worldscientific.com/doi/pdf/10.1142/S0129167X91000107>.
<http://www.worldscientific.com/doi/abs/10.1142/S0129167X91000107>.
- [19] H. Reckziegel, “On the eigenvalues of the shape operator of an isometric immersion into a space of constant curvature,” *Mathematische Annalen* **243** no. 1, (1979) 71–82.
<http://dx.doi.org/10.1007/BF01420208>.
- [20] U. Pinkall, “Curvature properties of taut submanifolds,” *Geometriae Dedicata* **20** no. 1, (1986) 79–83. <http://dx.doi.org/10.1007/BF00149273>.
- [21] U. Pinkall, “Dupin hypersurfaces,” *Mathematische Annalen* **270** no. 3, (1985) 427–440.
<http://dx.doi.org/10.1007/BF01473438>.
- [22] M. Dajczer, L. A. Florit, and R. Tojeiro, “Reducibility of Dupin submanifolds,” *Illinois J. Math.* **49** no. 3, (07, 2005) 759–791. <http://projecteuclid.org/euclid.ijm/1258138218>.
- [23] E. Cartan, “Sur des familles remarquables d’hypersurfaces isoparamétriques dans les espaces sphériques,” *Mathematische Zeitschrift* **45** no. 1, (1939) 335–367.
<http://dx.doi.org/10.1007/BF01580289>.
- [24] T. Levi-Civita, *Famiglie di superficie isoparametriche nel l’ordinario spazio euclideo*. Reale Accademia Nazionale, 1937. <https://books.google.it/books?id=Sg21YgEACAAJ>.
- [25] B. Segre, “Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni,” *Atti Accad. Naz. Lincei Rend. VI. Ser.* **27** (1938) 203–207.
- [26] É. Cartan, “Familles de surfaces isoparamétriques dans les espaces à courbure constante,” *Annali di Matematica Pura ed Applicata* **17** no. 1, (1938) 177–191.
<http://dx.doi.org/10.1007/BF02410700>.
- [27] S. Console and C. Olmos, “Clifford systems, algebraically constant second fundamental form and isoparametric hypersurfaces,” *manuscripta mathematica* **97** no. 3, (1998) 335–342.
<http://dx.doi.org/10.1007/s002290050106>.
- [28] P. Nurowski, “Distinguished dimensions for special Riemannian geometries,” *Journal of Geometry and Physics* **58** no. 9, (2008) 1148 – 1170.
<http://www.sciencedirect.com/science/article/pii/S0393044008000570>.
- [29] P. Nurowski, “Special Riemannian geometries modeled on distinguished symmetric spaces,” Tech. Rep. math.DG/0603663, Mar, 2006. <http://cds.cern.ch/record/939061>.
Contribution to the proceedings : Special geometries in mathematical physics, Kuhlungsborn, March 2006.
- [30] G. Machado de Freitas, “Submanifolds with homothetic Gauss map in codimension two,” *Geom. Dedicata* **180** (2016) 151–170. <http://dx.doi.org/10.1007/s10711-015-0096-2>.
- [31] U. Pinkall, “Dupin’sche Hyperflächen in E^4 ,” *manuscripta mathematica* **51** no. 1, (1985) 89–119.
<http://dx.doi.org/10.1007/BF01168348>.
- [32] E. T. Cecil and R. G. Jensen, “Dupin hypersurfaces with three principal curvatures,” *Inventiones mathematicae* **132** no. 1, (1998) 121–178.
<http://dx.doi.org/10.1007/s002220050220>.
- [33] T. E. Cecil and S.-S. Chern, “Tautness and Lie sphere geometry,” *Mathematische Annalen* **278** no. 1, (1987) 381–399.
<http://dx.doi.org/10.1007/BF01458076>.
- [34] A. C. Asperti and M. Dajczer, “ n -dimensional submanifolds of \mathbf{R}^{n+1} and S^{n+2} ,” *Illinois J. Math.* **28** no. 4, (1984) 621–645. <http://projecteuclid.org/euclid.ijm/1256045970>.
- [35] J. D. Moore, “Submanifolds of constant positive curvature I,” *Duke Math. J.* **44** no. 2, (06, 1977) 449–484.
<http://dx.doi.org/10.1215/S0012-7094-77-04421-0>.
- [36] S. Lie, “Über Komplexe, insbesondere Linien- und Kugelkomplexe, mit Anwendung auf der Theorie der partieller Differentialgleichungen,” *Math. Ann.* **5** (1872) 145–208, 209–256. (Ges. Abh. 2, 1–121).
- [37] S. Lie, *Geometrie der Berührungs transformationen*. Chelsea Publishing Co., New York, corrected ed., 1977. With editorial assistance by Georg Scheffers.

- [38] U. Pinkall, “Dupin’sche Hyperflächen,” *Dissertation, Universität Freiburg, Freiburg, Germany* (1981) .
- [39] T. E. Cecil, *Lie sphere geometry*. Universitext. Springer, New York, second ed., 2008. With applications to submanifolds.
- [40] U. Pinkall, “W-Kurven in der ebenen Lie-Geometrie. I.,” *Elemente der Mathematik* **39** (1984) 28–33. <http://eudml.org/doc/141333>.
- [41] U. Pinkall, “W-Kurven in der ebenen Lie-Geometrie. II.,” *Elemente der Mathematik* **39** (1984) 67–78. <http://eudml.org/doc/141339>.
- [42] P. Samuel, *Projective geometry*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1988. <http://dx.doi.org/10.1007/978-1-4612-3896-6>. Translated from the French by Silvio Levy, Readings in Mathematics.
- [43] D. H. Singley, “Smoothness theorems for the principal curvatures and principal vectors of a hypersurface,” *Rocky Mountain J. Math.* **5** no. 1, (03, 1975) 135–144. <http://dx.doi.org/10.1216/RMJ-1975-5-1-135>.
- [44] C. Dupin, “Applications de géométrie et de mécanique: À la marine, aux points et chaussées, etc.,” *Bachelier, Paris* (1822) .
- [45] J. Liouville, “Note au sujet de l’article précédent,” *J. Math. Pure Appl. (1)* **12** (1847) 265–290.
- [46] A. Cayley, “On the cyclide,” *Quart. J. Pure Appl. Math.* **12** (1873) 148–165. reprinted in *Collected Mathematical Papers*, Vol. 9, Cambridge University Press, Cambridge, UK, 1896, 64–78.
- [47] J. C. Maxwell, “On the cyclide,” *Quart. J. Pure Appl. Math.* **34** (1867) . reprinted in *Collected Works*, Vol. 2, Dover, New York, 1952, 144–159.
- [48] R. Lillienthal, “Besondere Flächen,” in *Encyklopädie der Mathematische Wissenschaften*, Vol. III, pp. 269–354. Teubner, Leipzig, 1902–1927.
- [49] T. F. Banchoff, “The spherical two-piece property and tight surfaces in spheres,” *J. Differential Geom.* **4** no. 2, (1970) 193–205. <http://projecteuclid.org/euclid.jdg/1214429383>.
- [50] T. E. Cecil, “Taut immersions of noncompact surfaces into a euclidean 3-space,” *J. Differential Geom.* **11** no. 3, (1976) 451–459. <http://projecteuclid.org/euclid.jdg/1214433599>.
- [51] F. Klein, *Vorlesungen über höhere Geometrie*. Dritte Auflage. Bearbeitet und herausgegeben von W. Blaschke. Die Grundlehren der mathematischen Wissenschaften, Band 22. Springer-Verlag, Berlin, 1968.
- [52] G. Darboux, *Leçons sur la théorie générale des surfaces. I, II*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1993. Généralités. Coordonnées curvilignes. Surfaces minima. [Generalities. Curvilinear coordinates. Minimum surfaces], Les congruences et les équations linéaires aux dérivées partielles. Les lignes tracées sur les surfaces. [Congruences and linear partial differential equations. Lines traced on surfaces], Reprint of the second (1914) edition (I) and the second (1915) edition (II), Cours de Géométrie de la Faculté des Sciences. [Course on Geometry of the Faculty of Science].
- [53] G. Darboux, *Leçons sur la théorie générale des surfaces. III, IV*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1993. Lignes géodésiques et courbure géodésique. Paramètres différentiels. Déformation des surfaces. [Geodesic lines and geodesic curvature. Differential parameters. Deformation of surfaces], Déformation infiniment petite et représentation sphérique. [Infinitely small deformation and spherical representation], Reprint of the 1894 original (III) and the 1896 original (IV), Cours de Géométrie de la Faculté des Sciences. [Course on Geometry of the Faculty of Science].
- [54] W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. Band I. Elementare Differentialgeometrie*. Dover Publications, New York, N. Y., 1945. 3d ed.
- [55] L. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*. Cornell University Library historical math monographs. Ginn, 1909. <https://books.google.it/books?id=hkENAAAYAAJ>.
- [56] D. Hilbert and S. Cohn-Vossen, *Geometry and the imagination*. Chelsea Publishing Company, New York, N. Y., 1952. Translated by P. Neményi.

- [57] K. Fladt and A. Baur, *Analytische Geometrie spezieller Flächen und Raumkurven*. Friedr. Vieweg & Sohn, Braunschweig, 1975. Sammlung Vieweg, Band 136.
- [58] T. E. Cecil and P. J. Ryan, “Conformal geometry and the cyclides of Dupin,” *Canad. J. Math.* **32** no. 4, (1980) 767–782. <http://dx.doi.org/10.4153/CJM-1980-059-1>.
- [59] T. E. Cecil and P. J. Ryan, *Tight and taut immersions of manifolds*, vol. 107 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [60] W. L. F. Degen, “Generalized cyclides for use in CAGD,” in *Computer-aided surface geometry and design (Bath, 1990)*, vol. 48 of *Inst. Math. Appl. Conf. Ser. New Ser.*, pp. 349–363. Oxford Univ. Press, New York, 1994.
- [61] M. J. Pratt, “Cyclides in computer aided geometric design,” *Comput. Aided Geom. Design* **7** no. 1-4, (1990) 221–242. [http://dx.doi.org/10.1016/0167-8396\(90\)90033-N](http://dx.doi.org/10.1016/0167-8396(90)90033-N). Curves and surfaces in CAGD ’89 (Oberwolfach, 1989).
- [62] M. J. Pratt, “Cyclides in computer aided geometric design. II,” *Comput. Aided Geom. Design* **12** no. 2, (1995) 131–152. [http://dx.doi.org/10.1016/0167-8396\(94\)00006-E](http://dx.doi.org/10.1016/0167-8396(94)00006-E).
- [63] Y. L. Srinivas and D. Dutta, “Blending and joining using cyclides,” *ASME Trans. J. Mechanical Design* **116** (1994) 1034–1041.
- [64] Y. L. Srinivas and D. Dutta, “Intuitive procedure for constructing geometrically complex objects using cyclides,” *CAD* **26** (1994) 327–335.
- [65] Y. L. Srinivas and D. Dutta, “Cyclides in geometric modeling: Computational tools for an algorithmic infrastructure,” *ASME Trans. J. Mechanical Design* **117** (1995) 363–373.
- [66] Y. L. Srinivas and D. Dutta, “Rational parametrization of parabolic cyclides,” *Comput. Aided Geom. Design* **12** (1995) 551–566.
- [67] M. Schrott and B. Odehnal, “Ortho-circles of Dupin cyclides,” *J. Geom. Graph.* **10** no. 1, (2006) 73–98.
- [68] T. E. Cecil and P. J. Ryan, “Focal sets, taut embeddings and the cyclides of dupin,” *Mathematische Annalen* **236** no. 2, (1978) 177–190. <http://dx.doi.org/10.1007/BF01351390>.
- [69] R. Tojeiro, “Conformal immersions of warped products,” *Geometriae Dedicata* **128** no. 1, (2007) 17–31. <http://dx.doi.org/10.1007/s10711-007-9179-z>.

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